Stability Analysis of a Plane Poisseuille Flow

Course Project: Water Waves and Hydrodynamic Stability 2016

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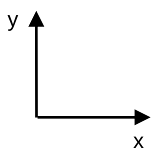
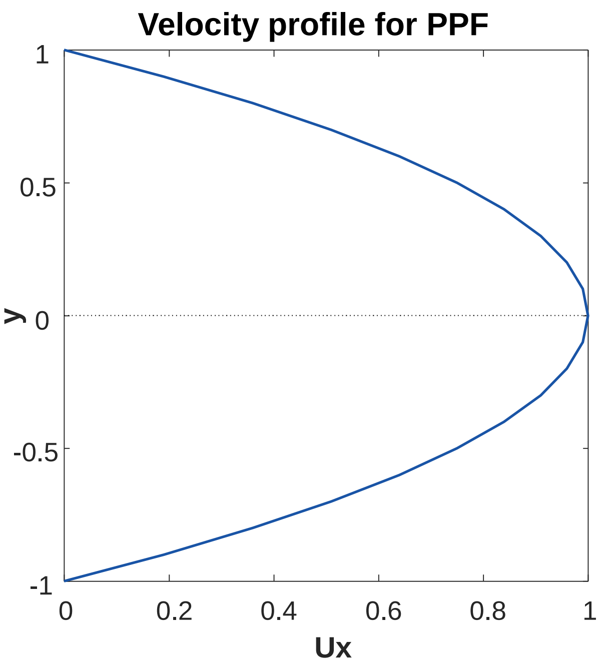
Plane Poiseuille Flow (PPF): Characteristics





Pressure-driven flow between two resting plates

Scaled velocity profile (only in y):

$$U(y) = \frac{U^*}{U_{Cl}} = 1 - y^2$$

No slip boundary conditions at wall:

$$U(y = \pm 1) = 0$$

Stability properties from literature:

$$Re_{cr,energy} = 49.6$$

$$Re_{cr,linear} = 5772$$

PPF: Governing Equations 1/3

Start with incompressible Navier-Stokes equation:

$$\frac{\partial u_i}{\partial t} = -u_j \cdot \frac{\partial u_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{Re} \cdot \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

- Introduce the disturbances: $\tilde{u}_i = U_{b,i} + u_i$, $\tilde{p} = P_b + p$
- Rearrange and linearize:

$$\frac{\partial u_i}{\partial t} + U_{b,j} \cdot \frac{\partial u_i}{\partial x_j} + u_j \cdot \frac{\partial U_{b,i}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{1}{Re} \cdot \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

PPF: Governing Equations 2/3

- Assume parallel flow: $U_{b,i} = U(y)\delta_{1i}$

$$\frac{\partial u}{\partial t} = -U_b \frac{\partial u}{\partial x} - U_b' v - \frac{\partial p}{\partial x} + \frac{1}{Re}\frac{\partial^2 u}{\partial x_j \partial x_j}$$
$$\frac{\partial v}{\partial t} = -U_b \frac{\partial v}{\partial x} - \frac{\partial p}{\partial y} + \frac{1}{Re}\frac{\partial^2 v}{\partial x_j \partial x_j}$$
$$\frac{\partial w}{\partial t} = -U_b \frac{\partial w}{\partial x} - \frac{\partial p}{\partial z} + \frac{1}{Re}\frac{\partial^2 w}{\partial x_j \partial x_j}$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

- Eliminate the pressure – divergence of momentum equation:

$$\frac{\partial^2 p}{\partial x_i \partial x_i} = -2U_b' \frac{\partial v}{\partial x}$$

PPF: Governing Equations 3/3

New BCs at walls:
$v = v' = \eta = 0$

Orr-Sommerfeld equation:

$$\frac{\partial v}{\partial t} = \left(-U_b \frac{\partial}{\partial x} \nabla^2 + U_b'' \frac{\partial}{\partial x} + \frac{1}{Re} \nabla^4 \right) v$$

Introducing $\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$ leads to the **squire equation:**

$$\frac{\partial \eta}{\partial t} = \left(-U_b \frac{\partial}{\partial x} + \frac{1}{Re} \nabla^2 \right) \eta - U_b' \frac{\partial v}{\partial z}$$

In matrix form:

external forcing

$$\frac{\partial}{\partial t} \begin{pmatrix} v \\ \eta \end{pmatrix} = \begin{pmatrix} L_{OS} & 0 \\ C & L_S \end{pmatrix} \begin{pmatrix} v \\ \eta \end{pmatrix} + B\boldsymbol{f}$$

Eigenvalue Stability Analysis

- Assume wavelike solutions: $v(x,y,z,t) = \hat{v}(y)e^{i(\alpha x+\beta z-\omega t)}$

$$-i\omega\begin{pmatrix} k^2 - D^2 & 0 \\ C & 1 \end{pmatrix}\begin{pmatrix} \hat{v} \\ \hat{\eta} \end{pmatrix} + \begin{pmatrix} L_{OS} & 0 \\ i\beta U_b' & L_S \end{pmatrix}\begin{pmatrix} \hat{v} \\ \hat{\eta} \end{pmatrix} = 0$$

with $L_{OS} = i\alpha U(k^2 - D^2) + i\alpha U_b'' + \frac{1}{Re}(k^2 - D^2)^2;\ L_{SQ} = i\alpha U + \frac{1}{Re}(k^2 - D^2)$

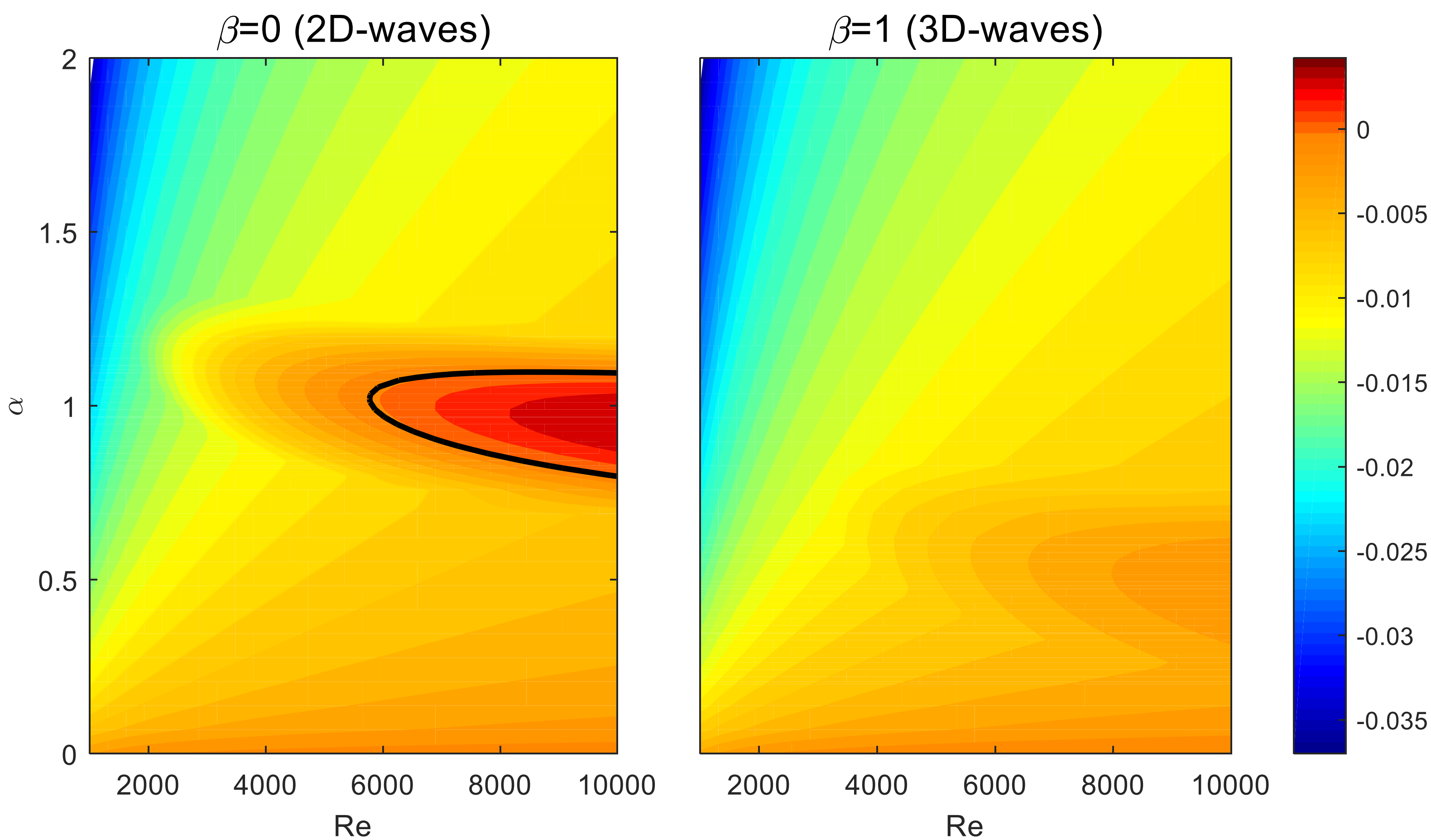
what leads to the EV problem:

$$(\boldsymbol{L} - i\omega\boldsymbol{M})\hat{\boldsymbol{q}} = 0$$

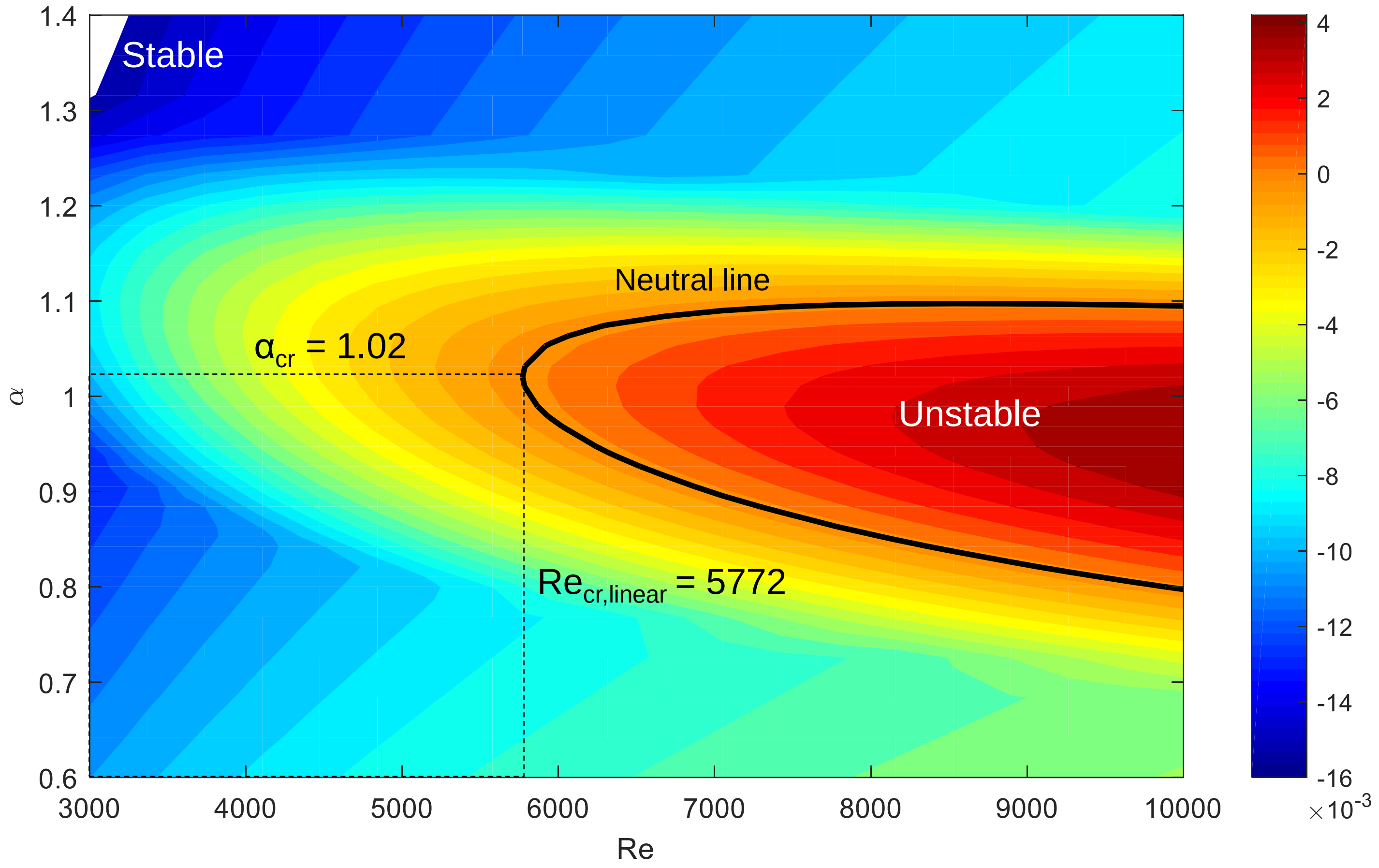
with the solution

$$\boldsymbol{q} = \boldsymbol{q_0}\exp(t\boldsymbol{L})$$

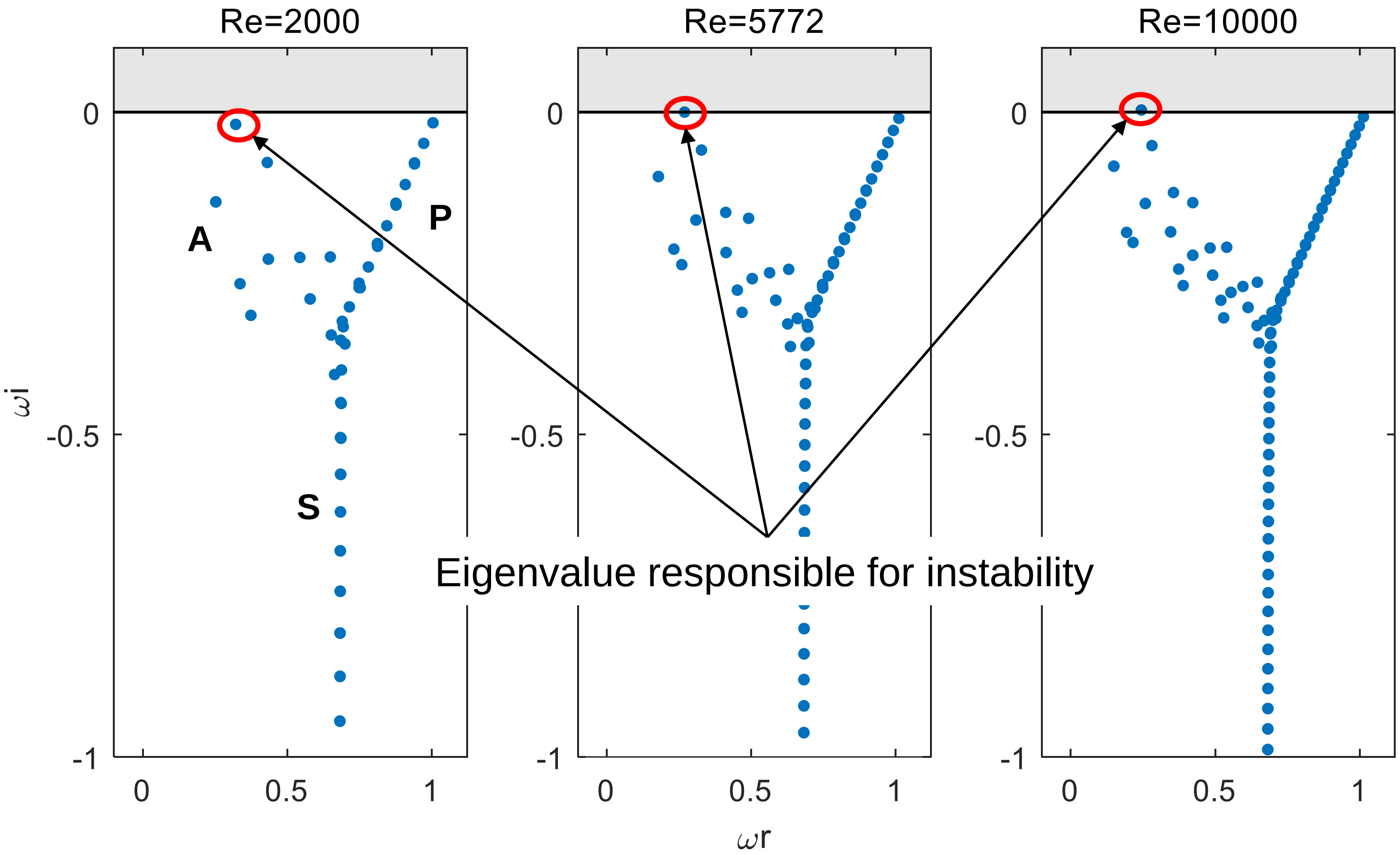
Long-time Stability: Maximum Eigenvalues



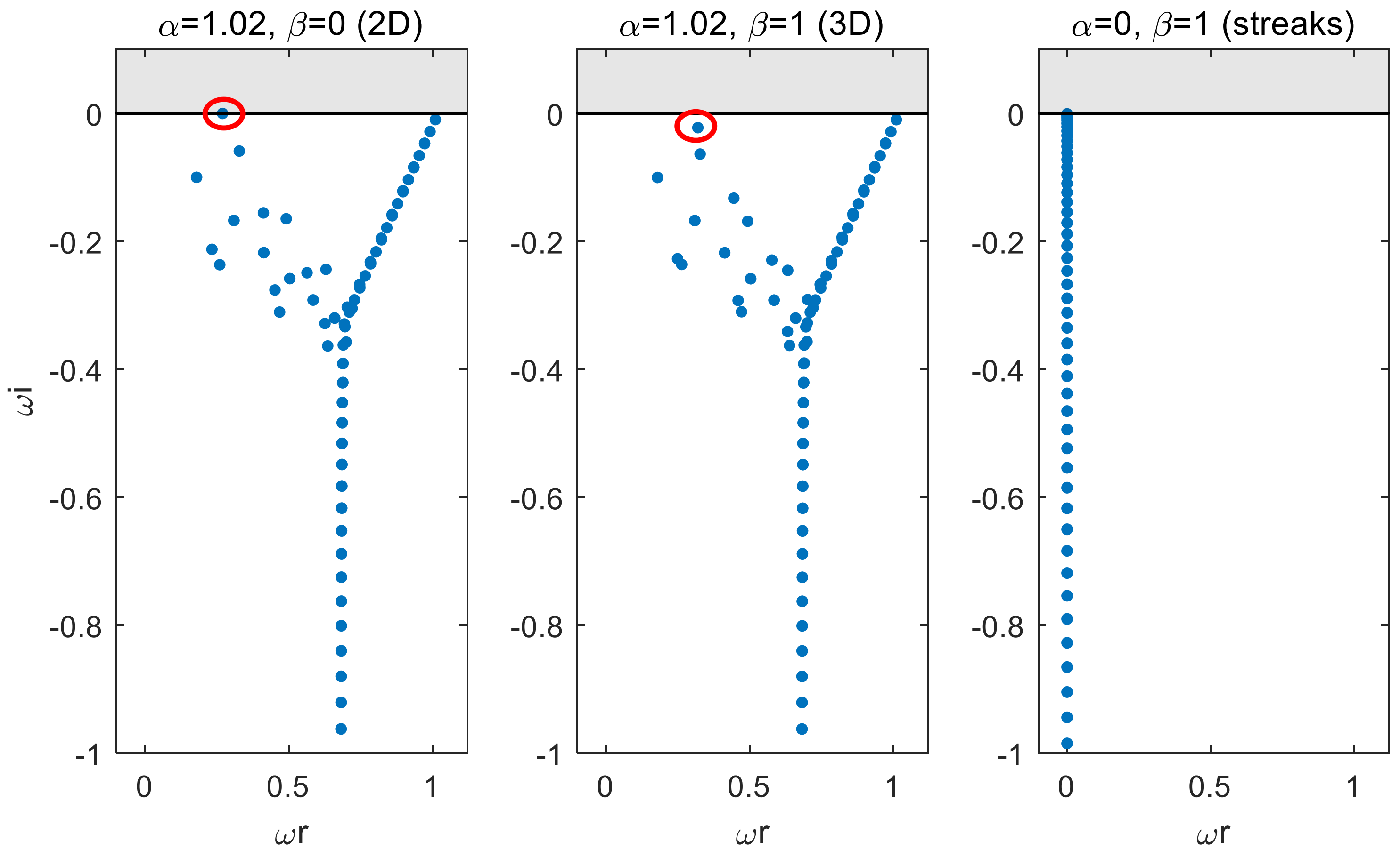
Long-time Stability: 2D-Waves



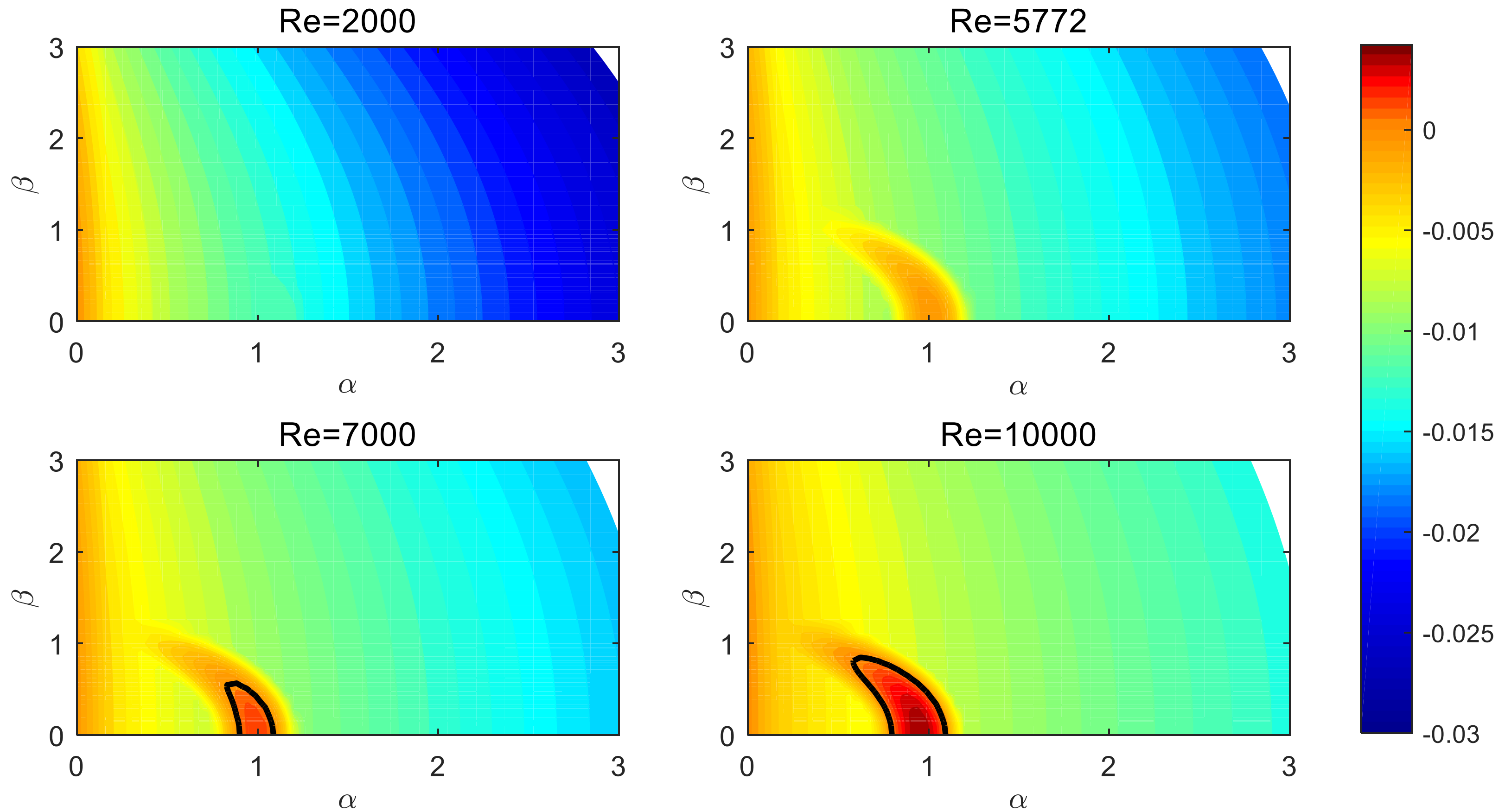
Eigenvalue Spectra: α=1.02, β=0



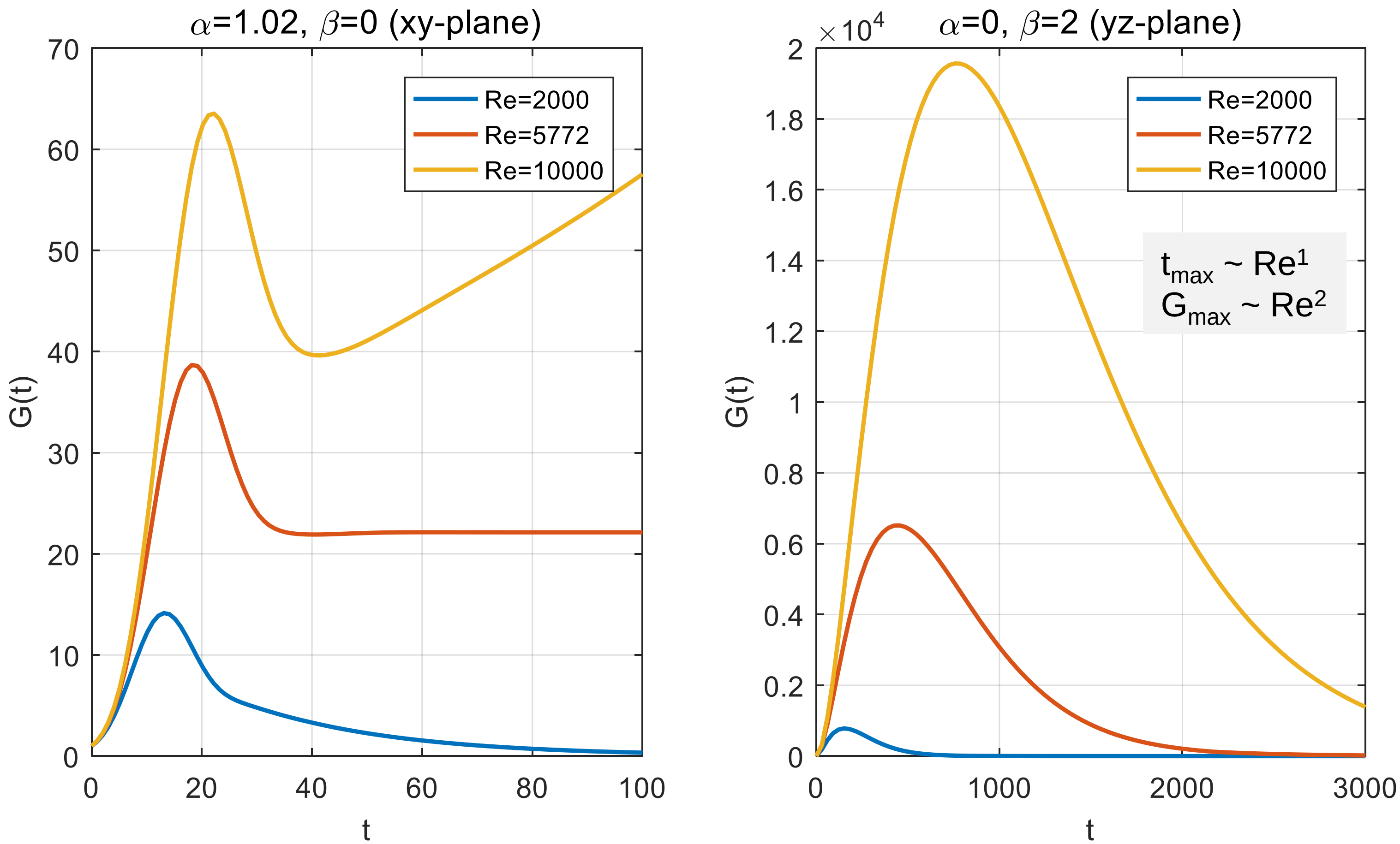
Eigenvalue Spectra: Re=5772



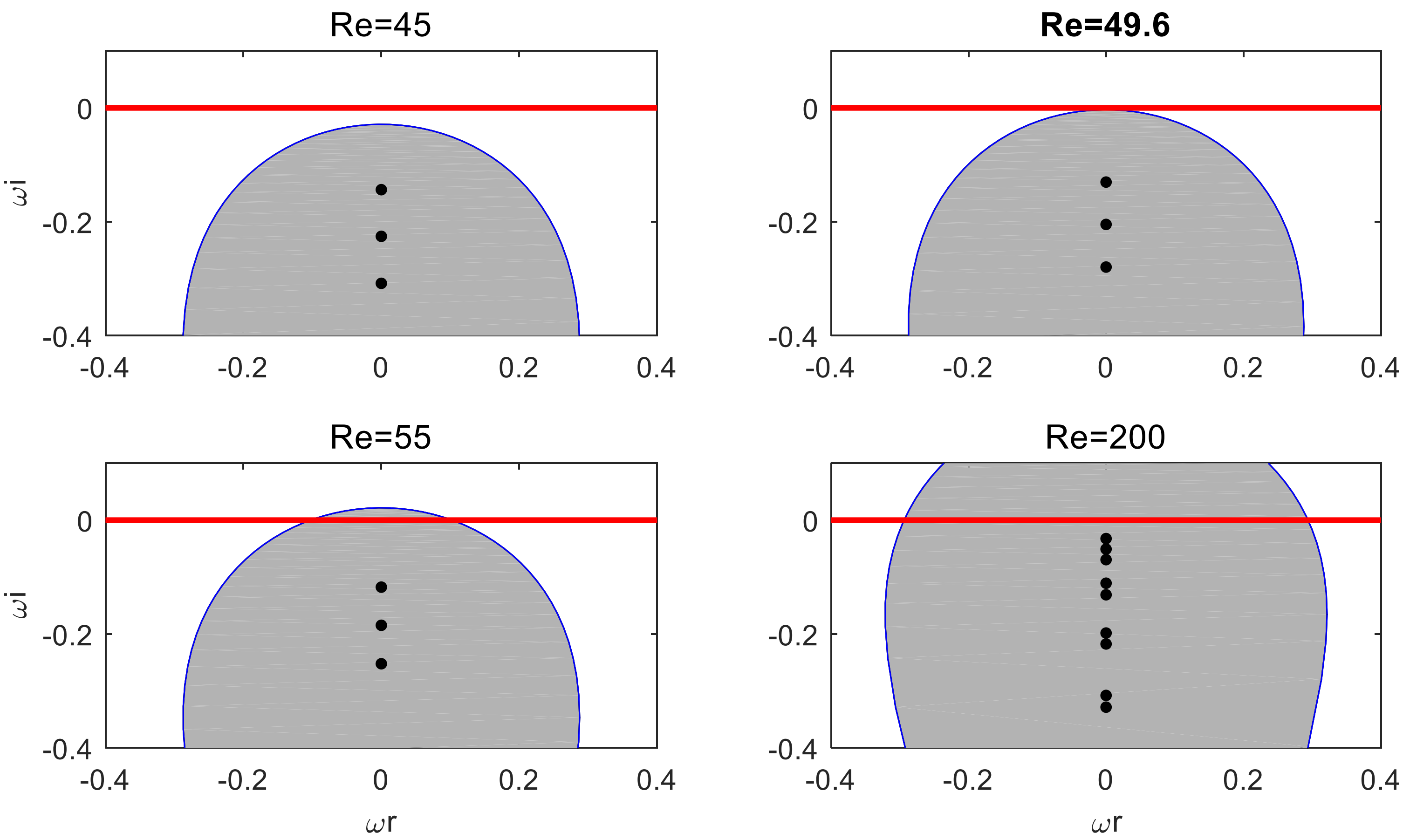
Eigenvalues: Summary



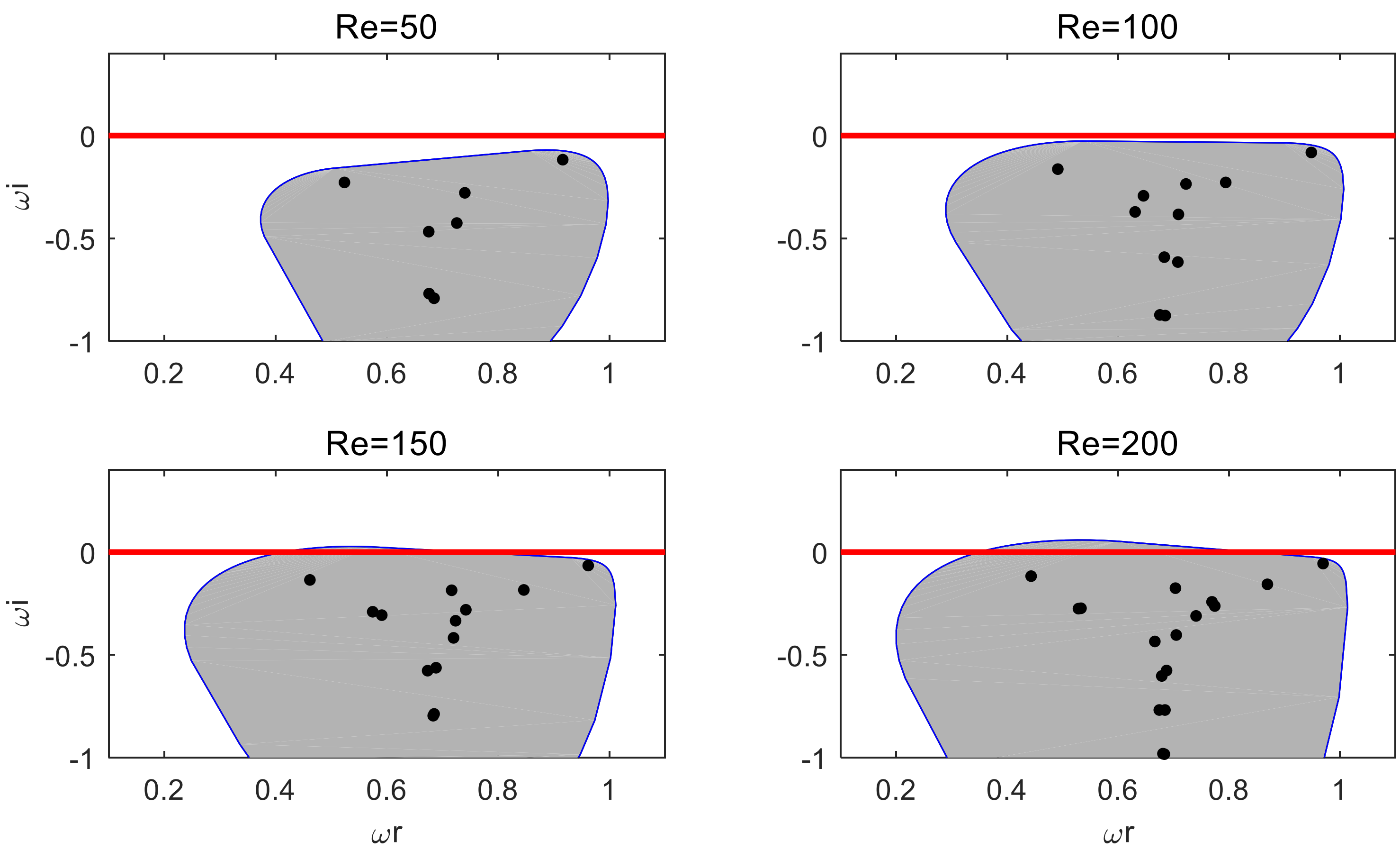
Transient Growth: G(t)



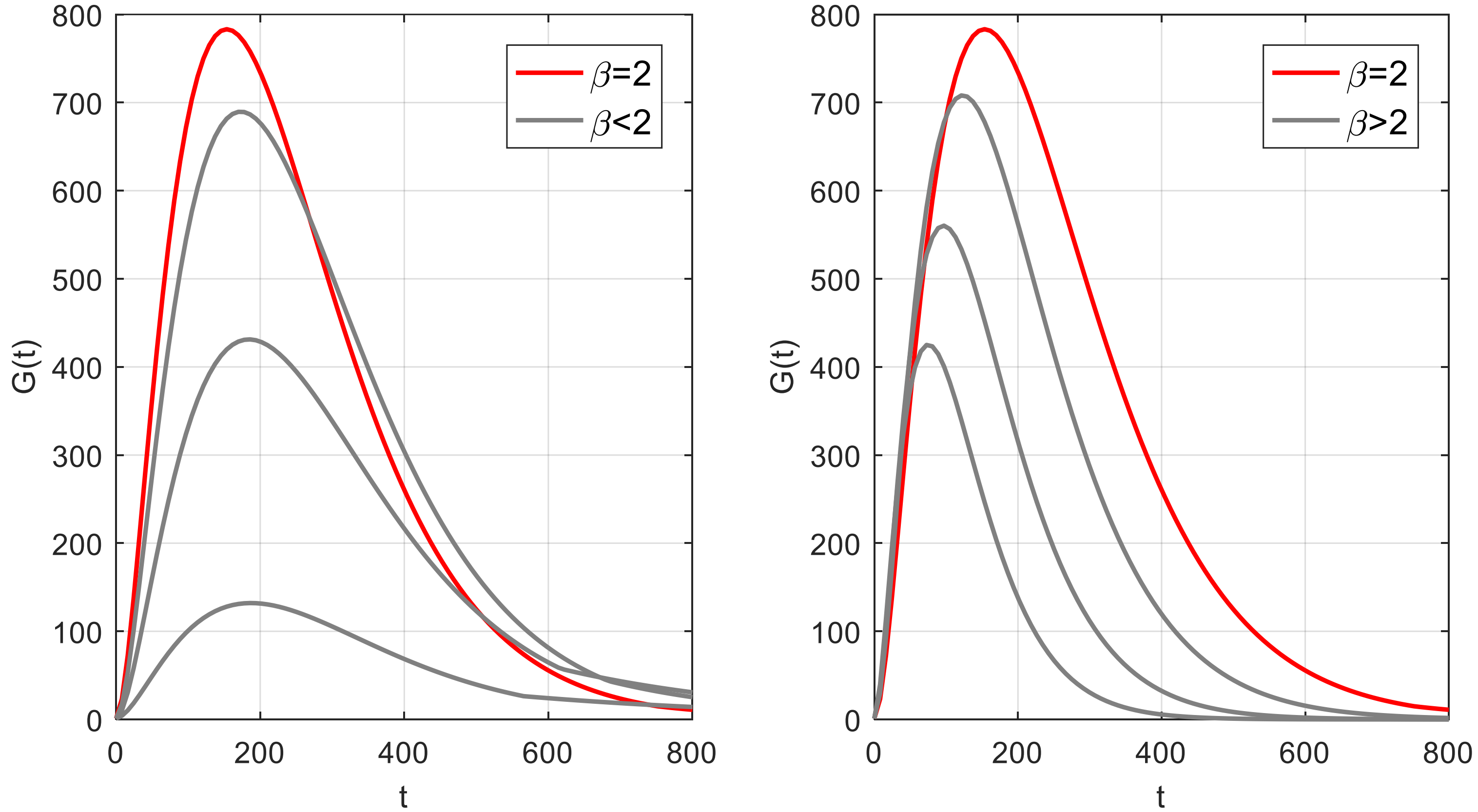
Numerical Abscissa: α=0, β=2



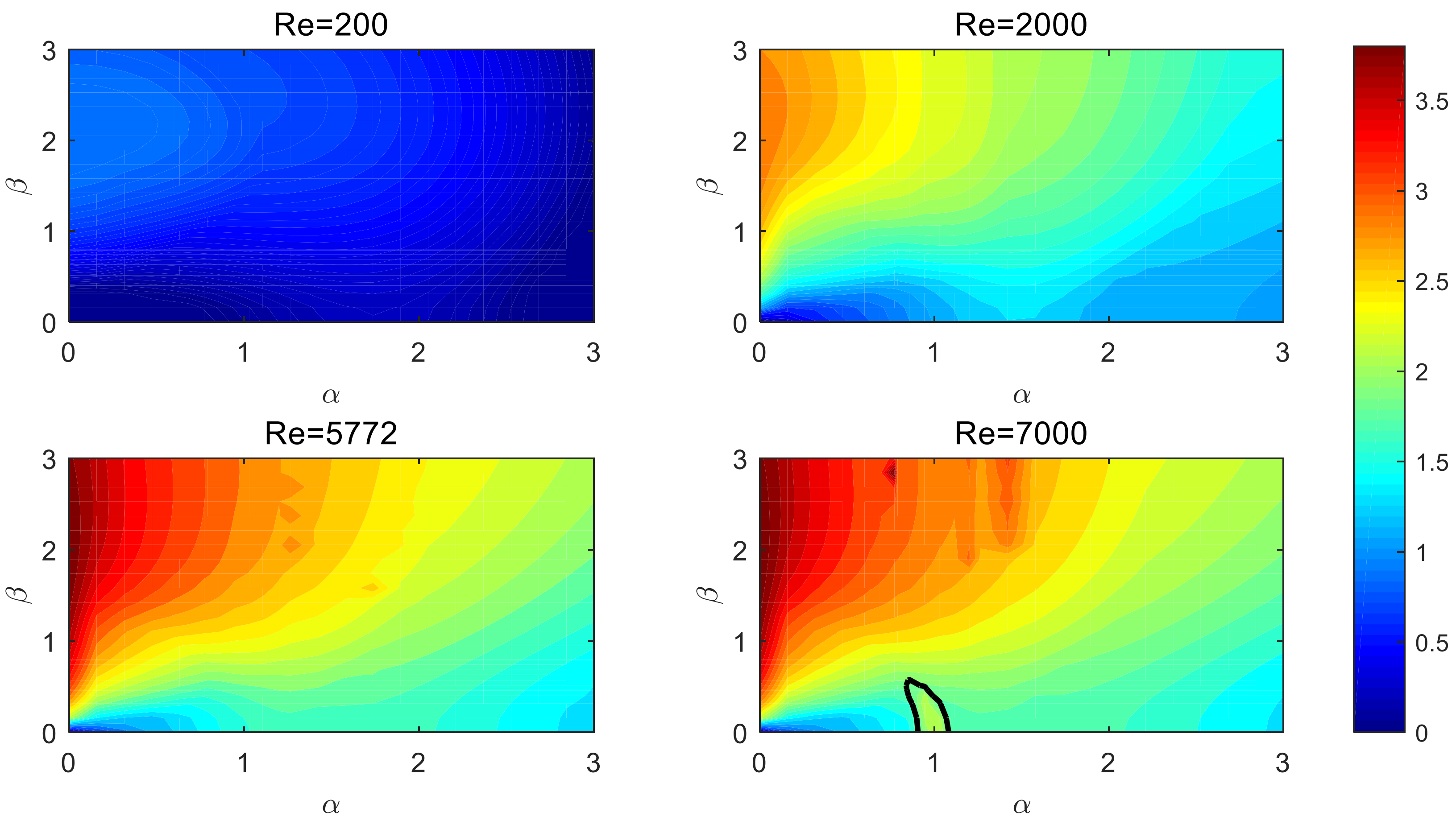
Numerical Abscissa: α=1.02, β=0



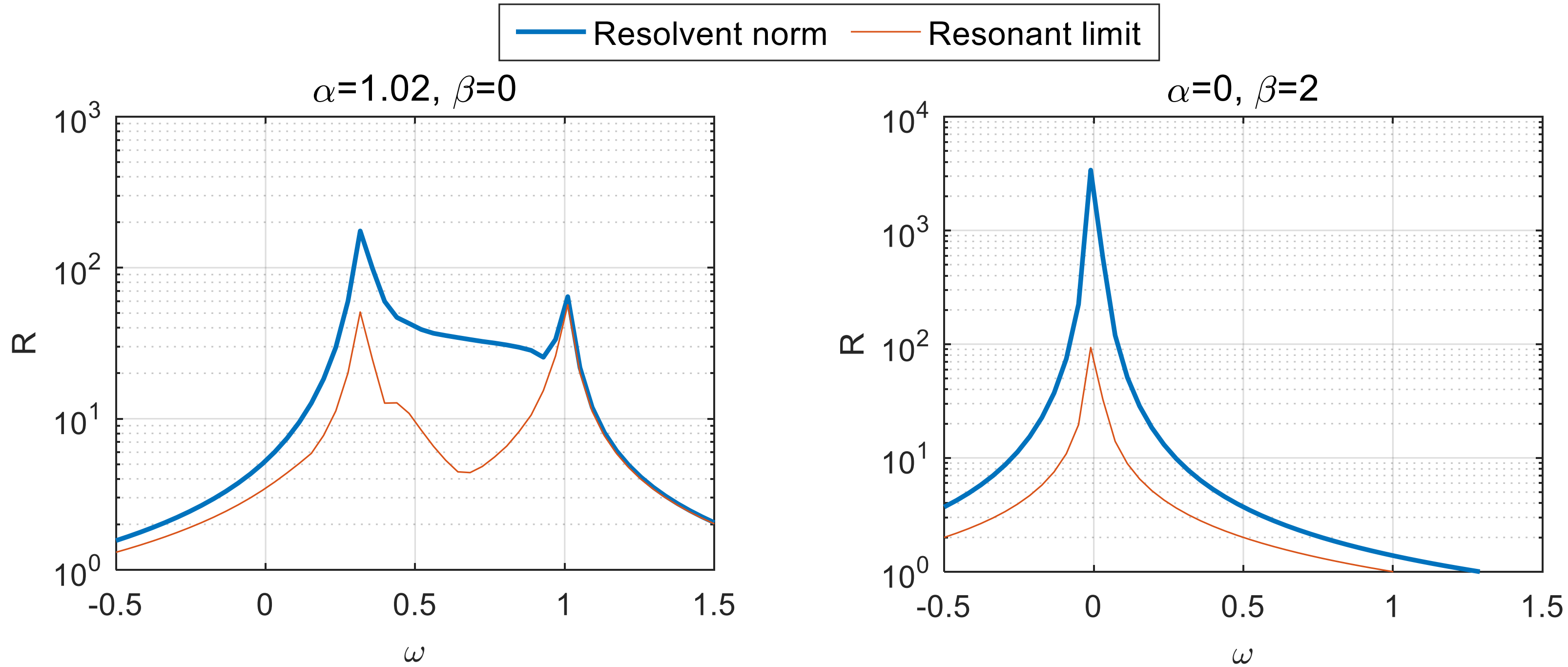
Transient Growth: Re=2000, α=0



Transient Growth: Summary



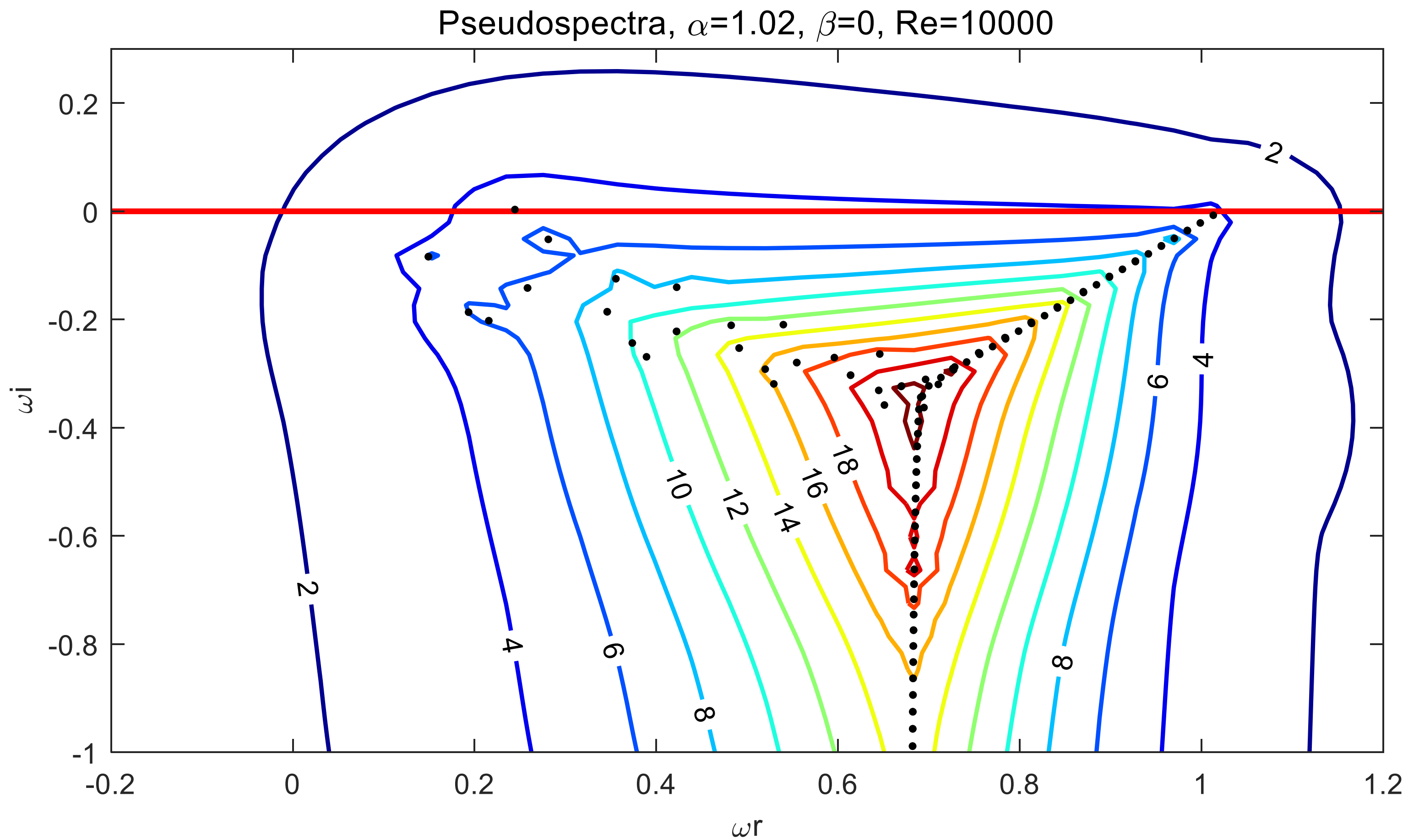
Resolvent Norm: Re=2000



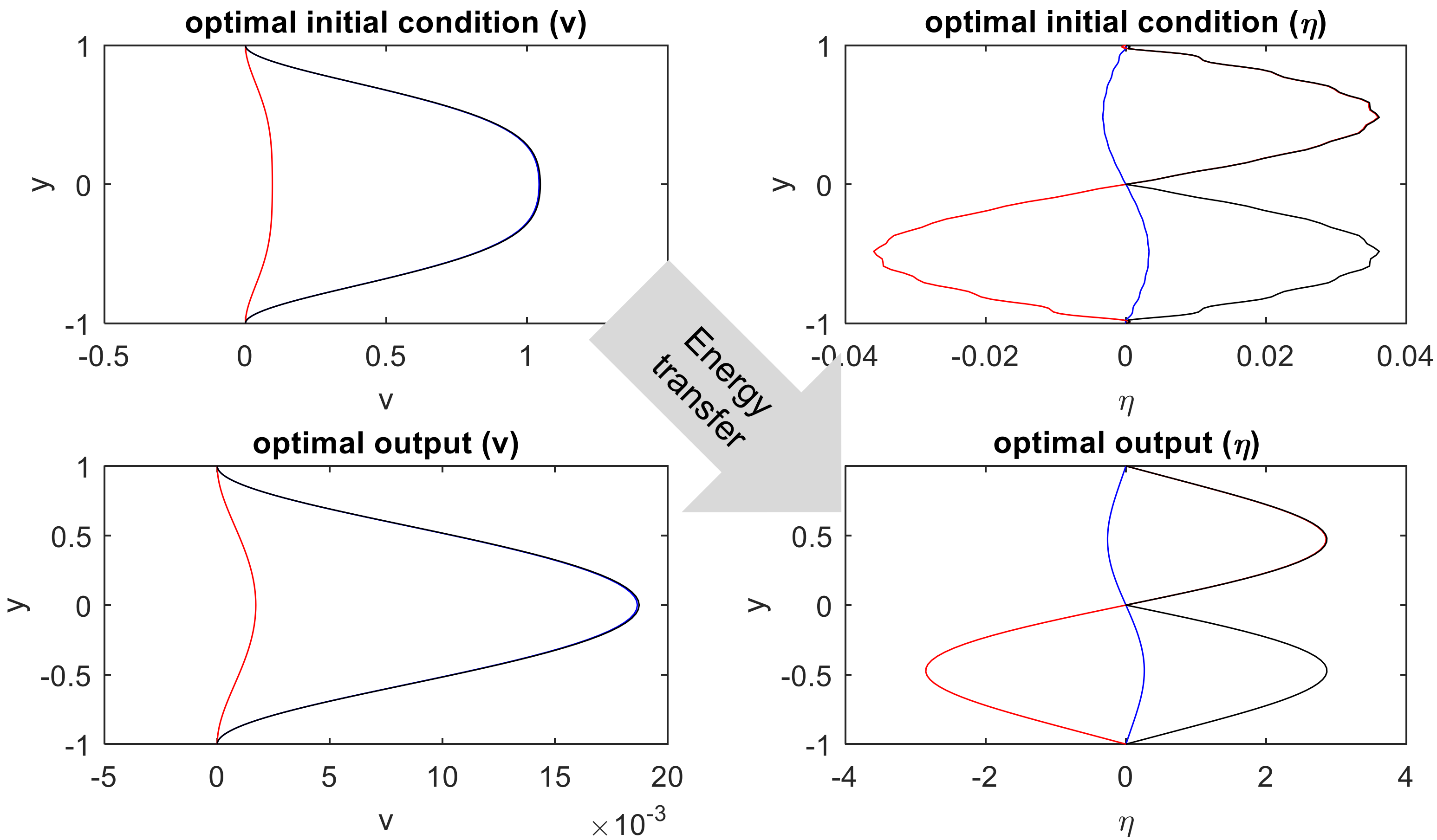
$$\frac{d}{dt}\boldsymbol{q} = L\boldsymbol{q} + \boldsymbol{f} \text{ with } \boldsymbol{f} = \hat{\boldsymbol{f}} exp(i\omega t)$$

$$R(\omega) = \max_{\hat{f}} \left(\frac{\|\hat{q}\|_E^2}{\|\hat{f}\|_E^2} \right) = \left\| (\boldsymbol{i\omega I} - \boldsymbol{L})^{-\boldsymbol{1}} \right\|_{\boldsymbol{E}}^{\boldsymbol{2}}$$

Pseudospectra



Optimal Disturbance: Re=2000, α=0, β=2



Conclusion

Eigenvalues:

- Onset of linear instability at $\alpha = 1.02$ and $Re = 5772$
- Instable eigenvalue in the A-branch
- P-,S-branch stays stable
- Different eigenvalue distribution for $\alpha=0$, no instability
- Eigenvalue responsible for instability relatively stable against disturbances

Transient Growth:

- Onset of transient growth at $Re=49.6$ at $\beta=2$ and $\alpha=0$
- Maximum amplification for $\beta=2$ and $\alpha=0$

Forcing and Optimal Response:

- For $\alpha=0$ only one peak in resolvent
- Two peaks for $\alpha\neq0$ due to eigenvalue distribution – distance
- Streaks: Energy transfer from v to η

Thank you for your attention!

Instability analysis

Long time Dynamics:

- Eigenvalue distribution: $L = S\mathbf{\Lambda}S^{-1}$
- Only maximum Eigenvalue
- Determines maximum growth for infinite time

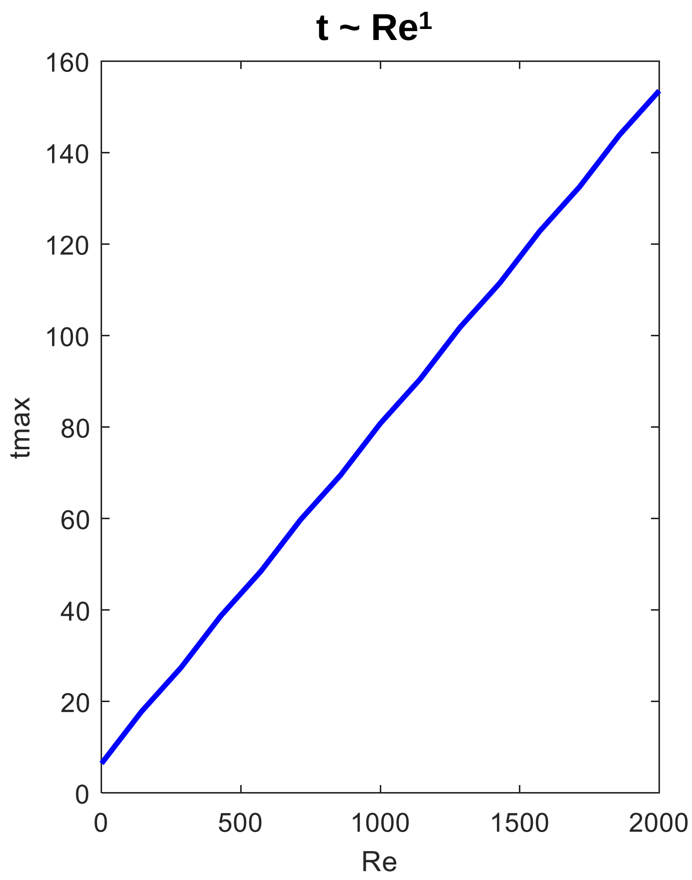
Short time Dynamics:

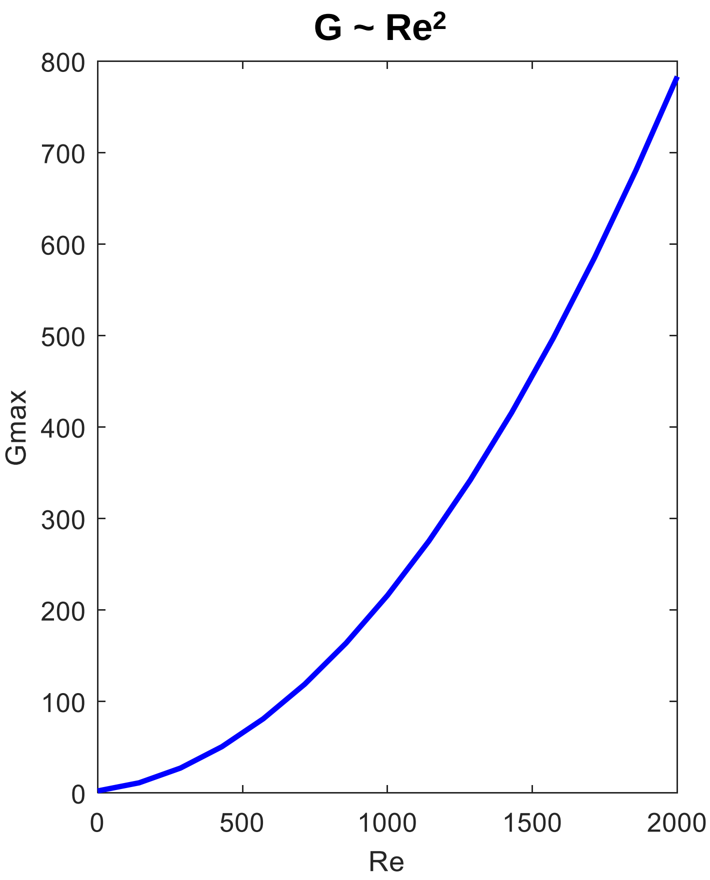
- Numerical Range: $\frac{\langle Lq,q\rangle}{\langle q,q\rangle}$

Combines both:

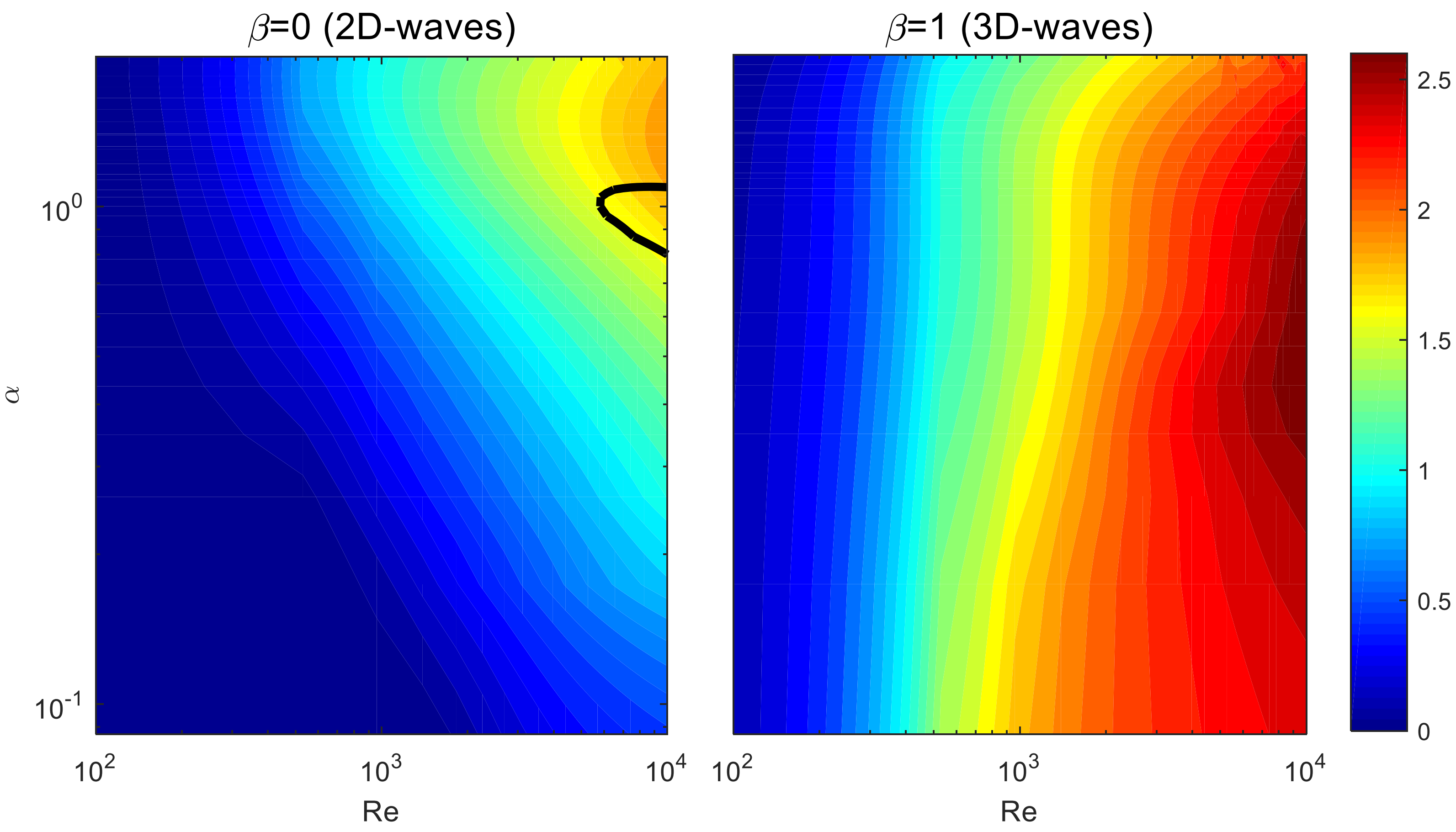
- Matrix exponential: $\|\exp(tL)\|_E^2$

Transient Growth: α=0, β=2

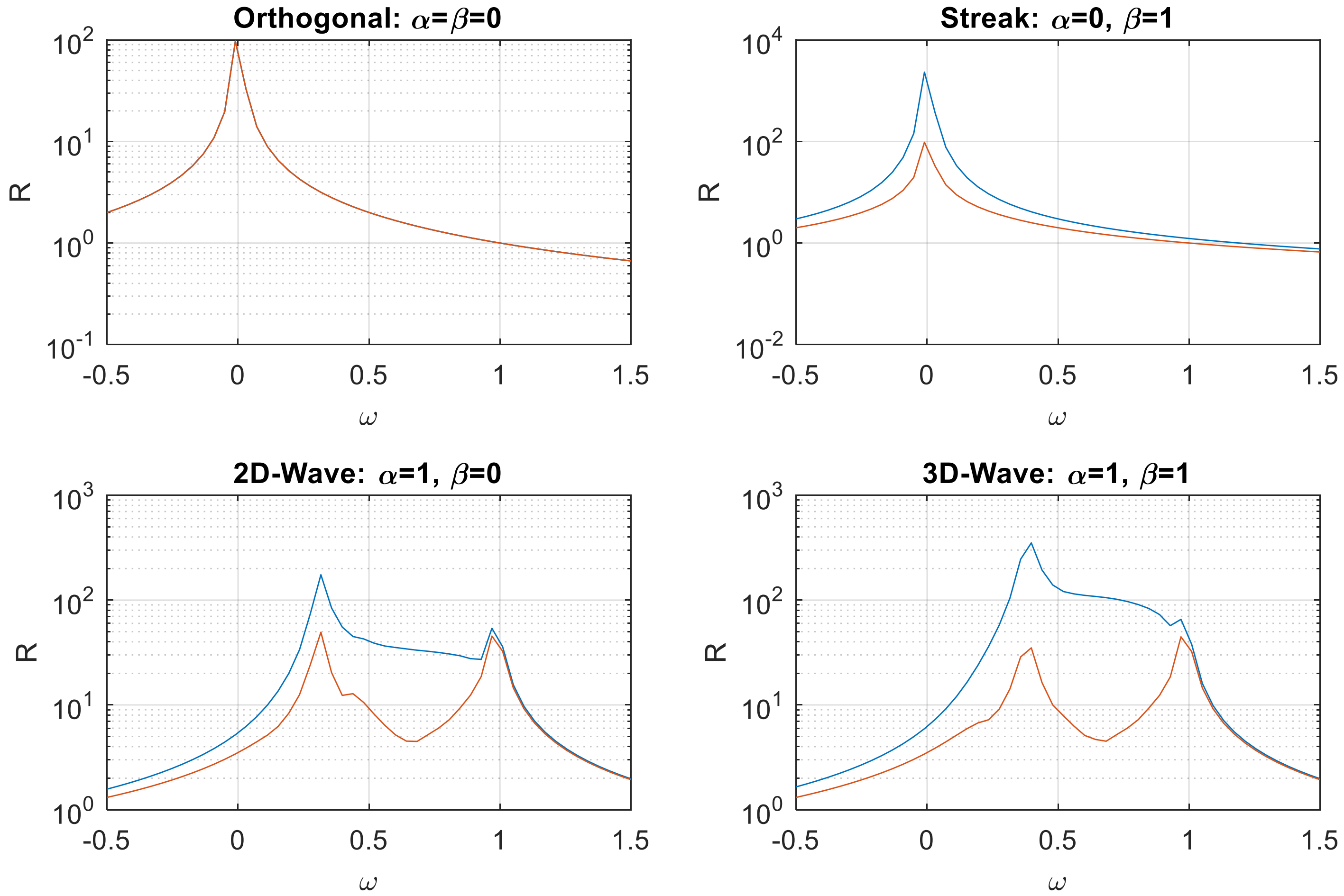




Transient Growth over α and Re



Resolvent Norm: Re=2000



Optimal Disturbance: Re=2000, α=1, β=0