SG2221 Wave Motion and Hydrodinamic Stability

MATLAB® Project on 2D Poiseuille Flow

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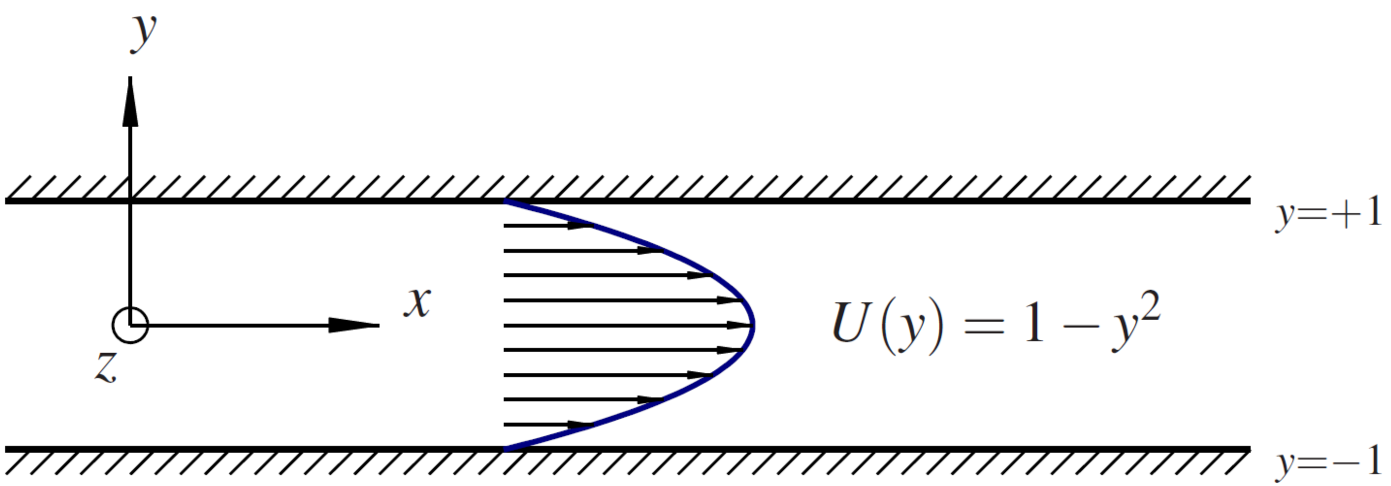
Base Flow



- 2D steady Incompressible Flow
- Flow driven by a constant pressure gradient
- Fully developed flow
- Normalized Velocity Profile
- No Slip boundary conditions

Governing Equations:

1. $\frac{\partial u_i}{\partial x_i} = 0$
2. $\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i$

With:

- $\frac{\partial}{\partial x} = \frac{\partial}{\partial t} = 0$
- $\frac{\partial p}{\partial x} = const < 0$
- $v(-1) = v(1) = u(-1) = u(1) = 0$

Disturbance Equations

- Reynolds decomposition of the Flow Field:

$$U + u \,;\, P + p$$

- Subtratct to the Navier Stokes equations the mean flow and neglect h.o.t. than linear:

$$\frac{\partial u_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial U_i}{\partial x_j} = -\frac{\partial p}{\partial x_j} + \frac{1}{Re} \nabla^2 u_j$$

↓

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + vU' = -\frac{\partial p}{\partial x} + \frac{1}{Re} \nabla^2 u$$

$$\frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} + \quad = -\frac{\partial p}{\partial y} + \frac{1}{Re} \nabla^2 v$$

$$\frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} + \quad = -\frac{\partial p}{\partial z} + \frac{1}{Re} \nabla^2 w$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Disturbance Equations, continuation

- Taking the divergence of the momentum equations, it yelds:

$$\nabla^2 p = -2U' \frac{\partial v}{\partial x}$$

Eliminating the pressure in the v-equation:

$$\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 - U''\frac{\partial}{\partial x} + \frac{1}{Re}\nabla^4\right] v = 0$$

Orr-Sommerfeld equation

- Afterwards the equation of the normal vorticity is considered to describe completely a 3D flow-field:

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$

Where η satisfies

$$\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) - \frac{1}{Re}\nabla^2\right]\eta = -U'\frac{\partial v}{\partial z}$$

Squire equation

Disturbance Equations, continuation

- The new system has to be completed with appropriate boundary conditions:

$$\eta\Big|_{walls} = v\Big|_{walls} = \frac{\partial v}{\partial y}\Big|_{walls} = 0$$

- And initial conditions:

$$u(x, y, z, 0) = u_0(x, y, z)$$
$$v(x, y, z, 0) = v_0(x, y, z)$$
$$w(x, y, z, 0) = w_0(x, y, z)$$

$$\downarrow$$

$$v(x, y, z, 0) = v_0(x, y, x)$$
$$\eta(x, y, z, 0) = \eta_0(x, y, z)$$

Orr-Sommerfeld and Squire Equations

Assuming a wave like solution of the form $q(x, y, z, t) = \tilde{q}e^{i(\alpha x+\beta z-\omega t)}$, the derived set of equation reads:

$$\left[(-i\omega + i\alpha U)(D^2 - k^2) - i\alpha U'' - \frac{1}{Re}(D^2 - k^2)^2\right]\tilde{v} = 0$$

$$\left[(-i\omega + i\alpha U) - \frac{1}{Re}(D^2 - k^2)\right]\tilde{\eta} = -i\beta U'\tilde{v}$$

and can be solved for the Orr.Sommerfeld modes and Squire modes independently.

Squire's Theorem

Thanks to Squire's theorem it is possible to conclude that parallel shear flow first become unstable for 2D wavelike perturbations at a value of Re number that is smaller than any value for which unstable 3D perturbation exists.

$$Re_c \triangleq \min_{\alpha,\beta} Re_L(\alpha,\beta) = \min_{\alpha} Re_L(\alpha,0)$$

A proof of the theorem can be derived using Squire's transformation in the perturbed equations of motion → $\omega = \alpha c$, and comparing with the case of $\beta = 0$. Comparing the eqution they have the same solution if the following equalities hold:

$$\alpha_{2D} = k = \sqrt{\alpha^2 + \beta^2}$$

$$\alpha_{2D} Re_{2D} = \alpha Re$$

↓

$$Re_{2D} = Re\frac{\alpha}{k} < Re$$

Matrix Formulation of the EIGV Problem

$$\frac{\partial}{\partial t}\begin{pmatrix} -D^2+k^2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \tilde{v} \\ \tilde{\eta} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{OS} & 0 \\ -i\beta U' & \mathcal{L}_{SQ} \end{pmatrix}\begin{pmatrix} \tilde{v} \\ \tilde{\eta} \end{pmatrix}$$

$$\mathcal{L}_{OS} = -i\alpha U(k^2-D^2) - i\alpha U'' - \frac{1}{Re}(k^2-D^2)^2$$

$$\mathcal{L}_{SQ} = -i\alpha U - \frac{1}{Re}(k^2-D^2)$$

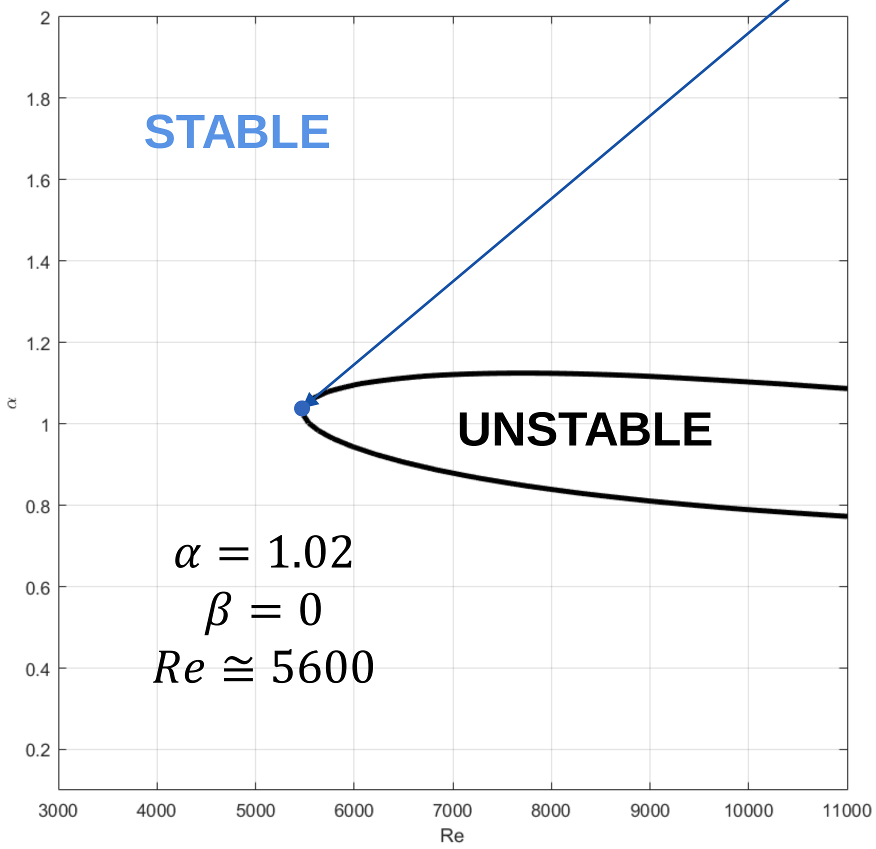
$$\frac{\partial}{\partial t}\boldsymbol{Mq} = \boldsymbol{Lq}$$

$$\frac{\partial}{\partial t}\boldsymbol{q} = \boldsymbol{M}^{-1}\boldsymbol{Lq} = \boldsymbol{L}_1\boldsymbol{q}$$

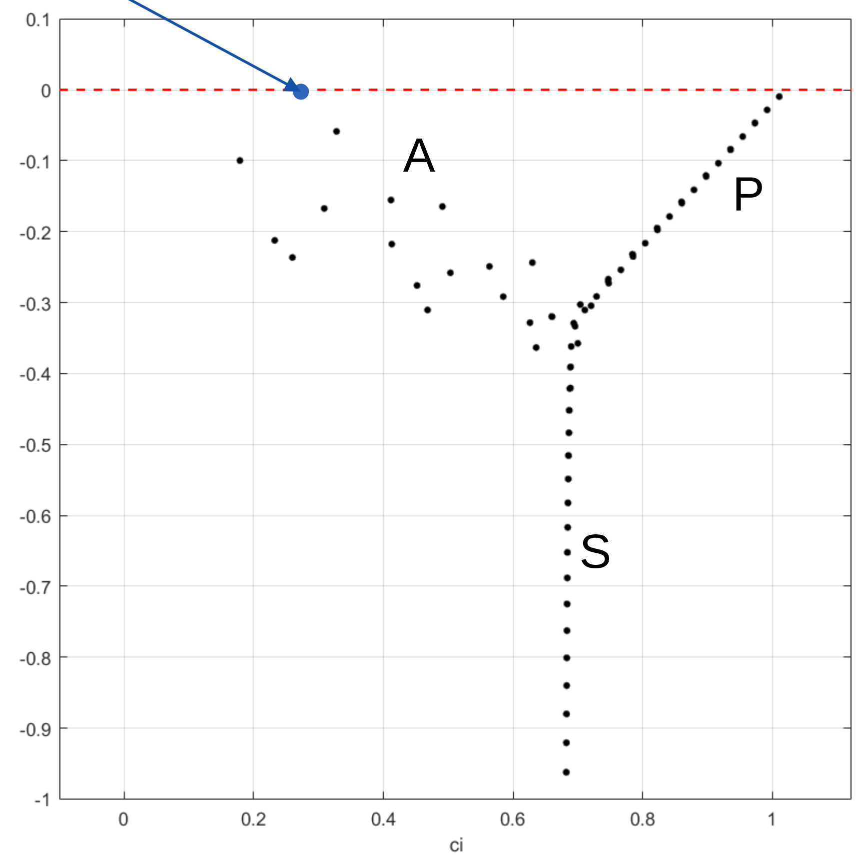
NOTE: the different OS and SQ eigenmodes can be found independently due to the triangular form of L_1

Neutral Curve and Spectrum

Neutral curve $c_i = 0$



Spectrum



Stability Analysis and Energy Growth

Traditional Lyapunov stability analysis based on infinite time horizon might not be interesting for our study due to the lack of representation of the «transient growth» that can occur for non modal system. We will look at stability as the amplification of initial energy perturbation over a period of time. The amplifiction depends on initial conditions and this dependence can be eliminated by optimizing over all permissible initial conditions, and accepting the maximum as the optimal energy amplification.

$$G(t) = \max_{\boldsymbol{q_0}} \frac{\|\mathbf{q}(\mathrm{t})\|_E^2}{\|\mathbf{q_0}\|_E^2} = \|\exp(\boldsymbol{L}t)\|_E^2$$

The energy norm of the matrix exponential is thus the largest amplification of energy any initial perturbation can experience over a given time interval.

It is important to realize that the energy amplification G(t) is optimal over all possible initial conditions, but that for each chosen time span t a different initial condition may yield the optimal gain G(t). The curve G(t) versus t may thus be thought of as an envelope over optimal initial conditions.

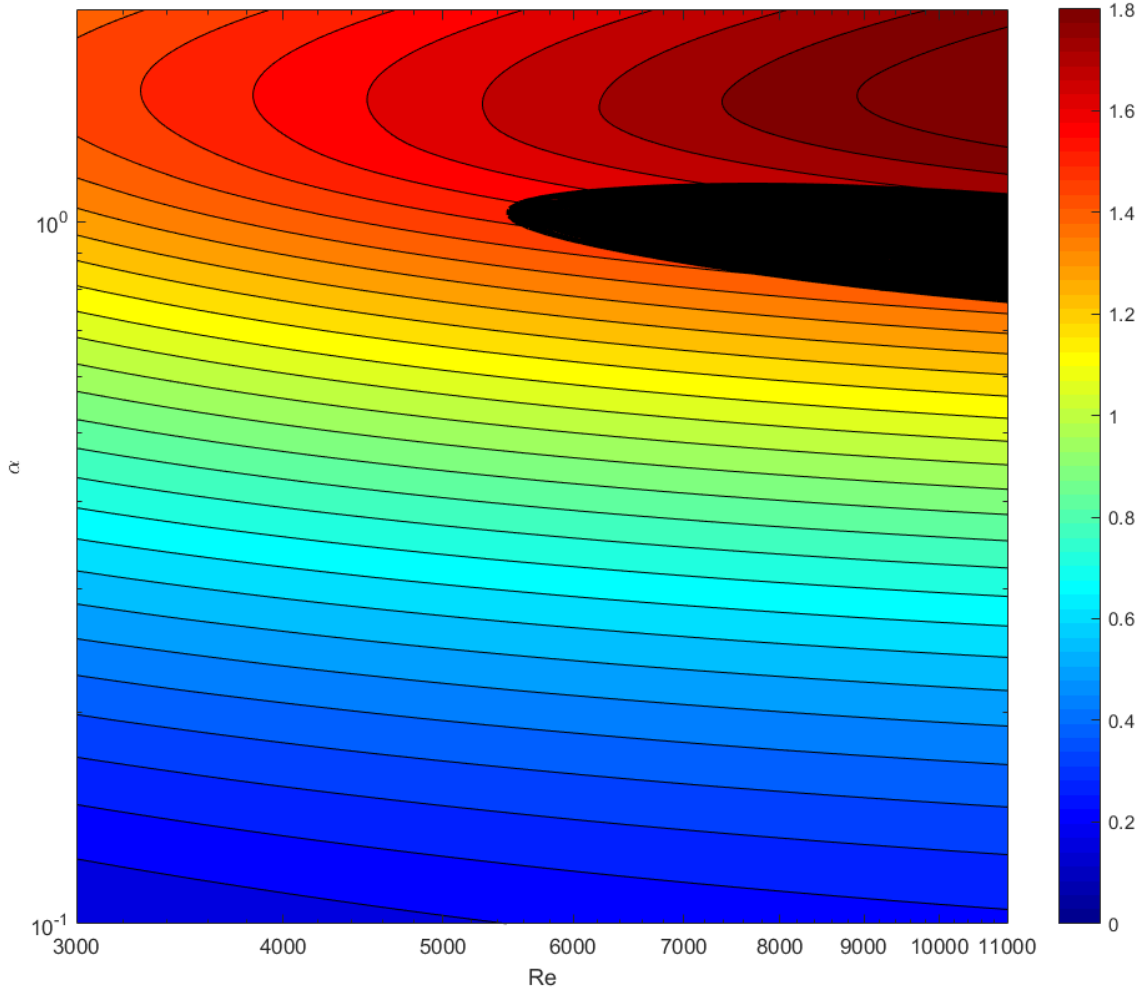
Contours of Constant Growth Rate $\log10 G_{max}$

$Re \in [3000\,;11000]$

$\alpha \in [0.1\,;2]$

$\beta = 0$

$T \in [0.\,;15]\ s$



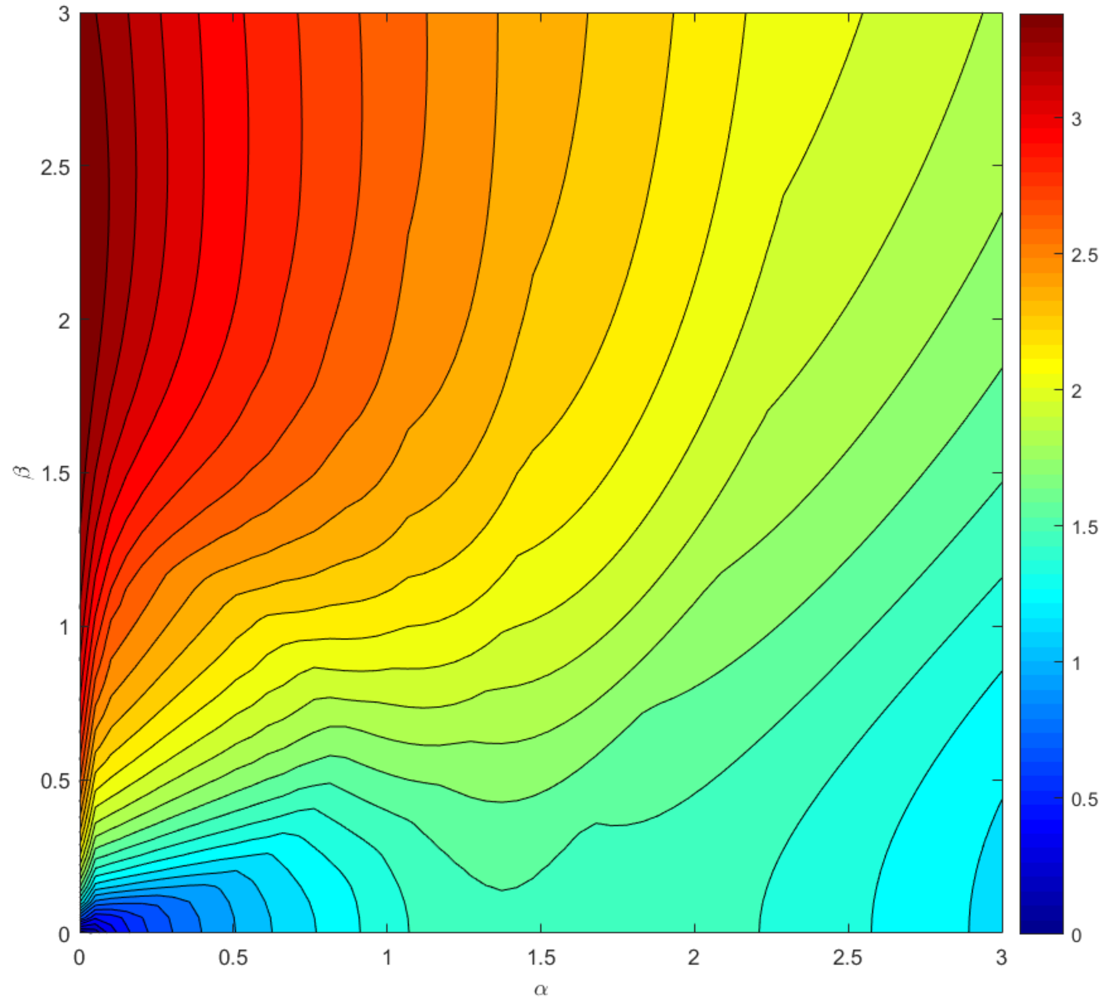
Contours of Constant Growth Rate $\log10 G_{max}$

$\alpha \in [0\,;3]$

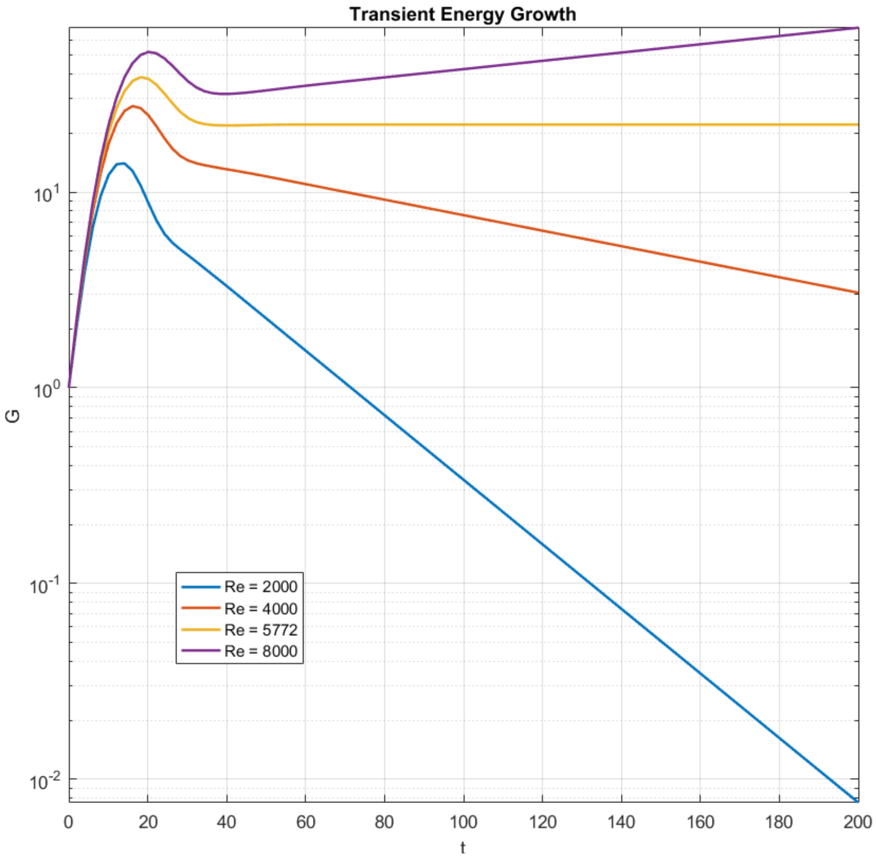
$\beta \in [0\,;3]$

$Re = 4000$

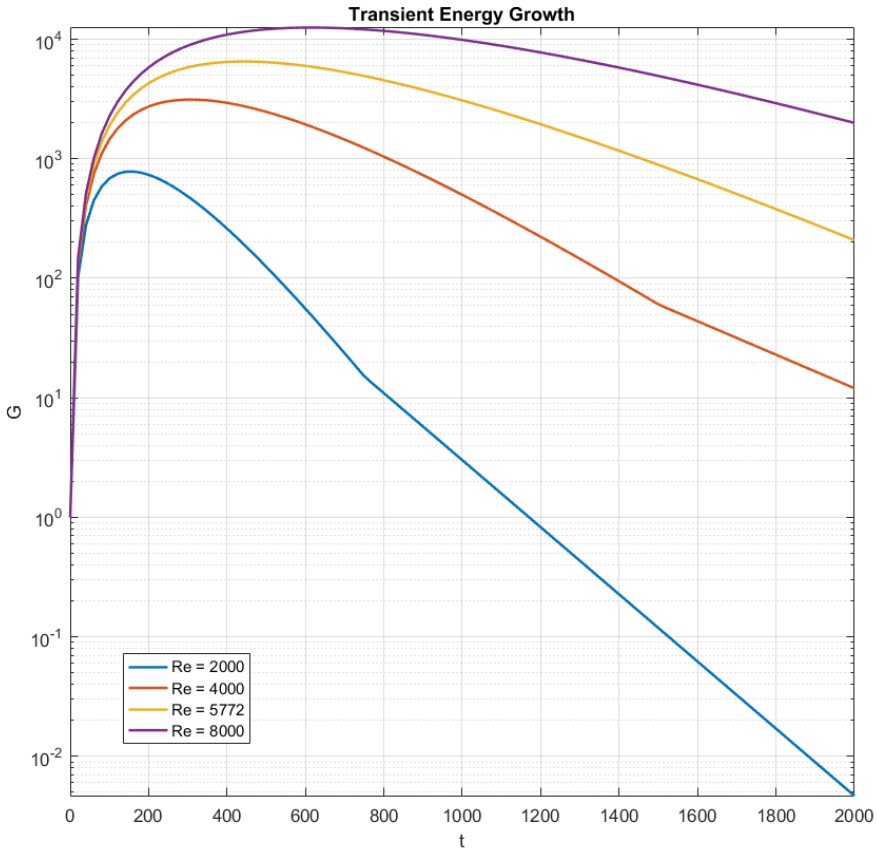
$T \in [0;500]\ s$



Transient Growth



$\alpha = 1.02; \beta = 0$



$\alpha = 0; \beta = 2$

Numerical abscissa and Numerical Range

The numerical abscissa is useful to analyze the short time dynamics of a system without computing the whole exponential matrix:

$$\left.\frac{dG}{dt}\right|_{0^+} = \frac{\langle \boldsymbol{q_0}, (\boldsymbol{L} + \boldsymbol{L^H})\boldsymbol{q_0} \rangle}{\langle \boldsymbol{q_0}, \boldsymbol{q_0} \rangle} = \lambda_{max}(\boldsymbol{L} + \boldsymbol{L^H}),$$

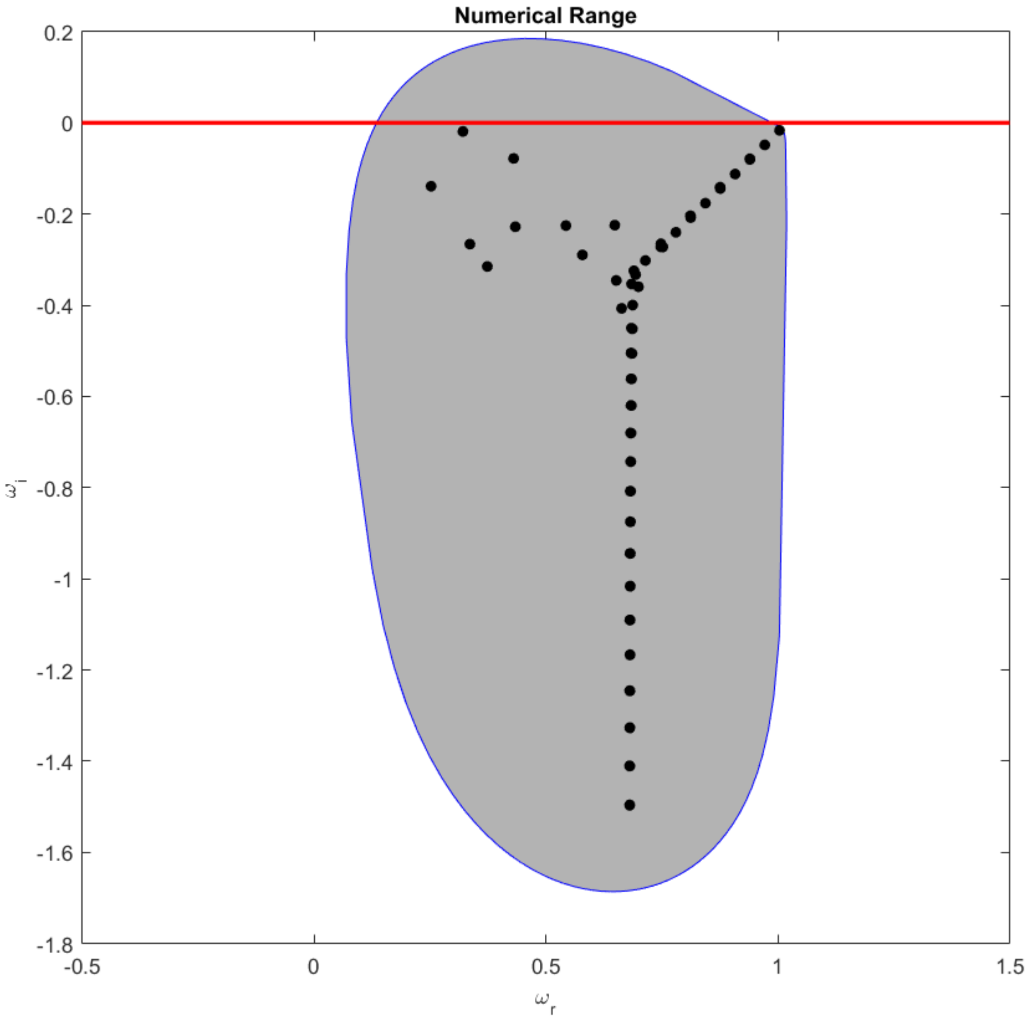
That is the initial slope of the gain curve.
The numerical abscissa can be generalized to the concept of numerical range. Considering the energy growth rate we have:

$$\frac{1}{E}\frac{dE}{dt} = 2\,Real\left(\frac{\langle \boldsymbol{Lq}, \boldsymbol{q} \rangle_E}{\langle \boldsymbol{q}, \boldsymbol{q} \rangle_E}\right).$$

The last expression establishes a link between the energy growth rate G(t) and the set of all Rayleigh quotients.

Numerical Range

- Transient growth is expected due to the protusion of the numerical range in the positive region for ω_i.
- The numerical range contains the spectrum of L.
- It is convex.
- The system is non-normal (the numerical range is not the convex hull of the spectrum)

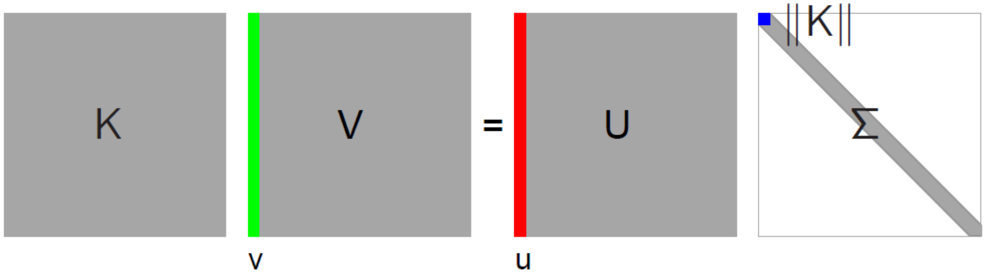


Optimal Initial Condition

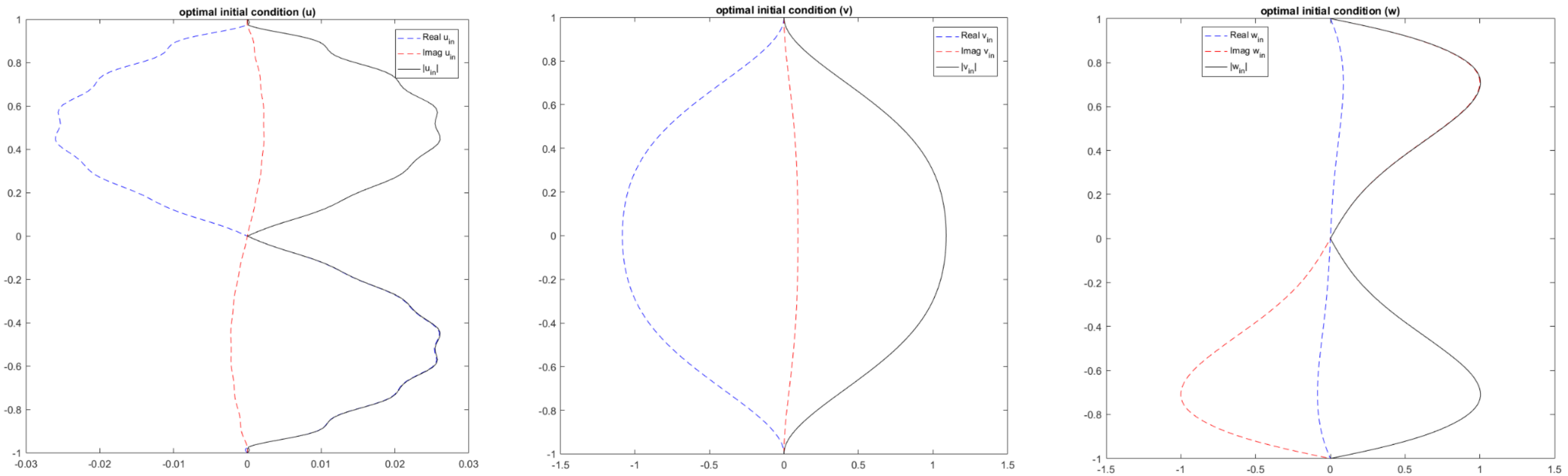
Sometimes it is interesting to calculate the optimal initial conditions that realizes the maximum amplification of G. Recalling the definition of the matrix exponential norm, it is given by an optimization over all initial conditions; the resulting curve G(t) is thus an envelope over many individual realizations. For the initial condition that yields to the maximum energy amplification we can write:

$$\exp(\boldsymbol{L}t^*)\boldsymbol{q_0} = \|\exp(\boldsymbol{L}t^*)\|\boldsymbol{q}$$

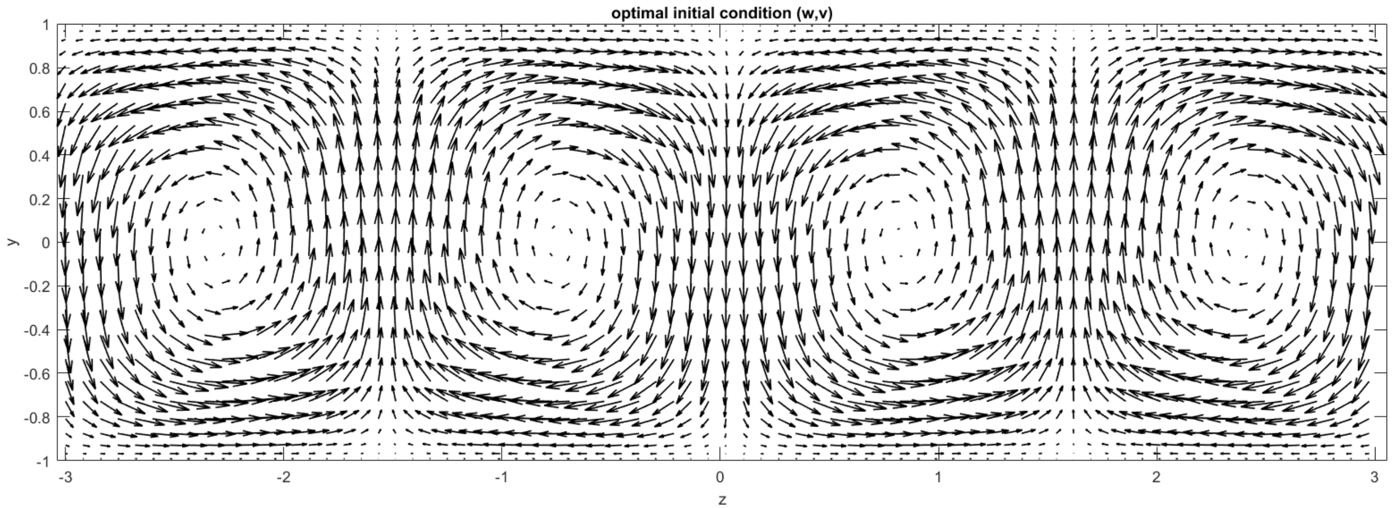
Through the singular value decomposition is then possible to extract the principal components and to recover the optimal initial condition.

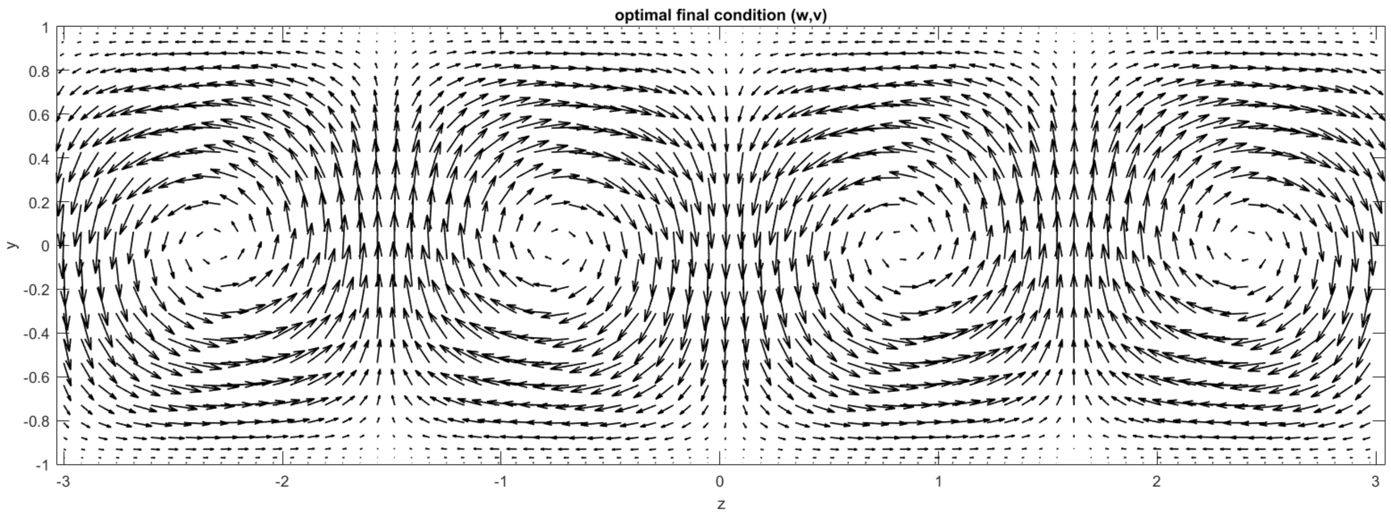


Optimal Initial Condition $(\alpha = 0, \beta = 2, Re = 1000)$

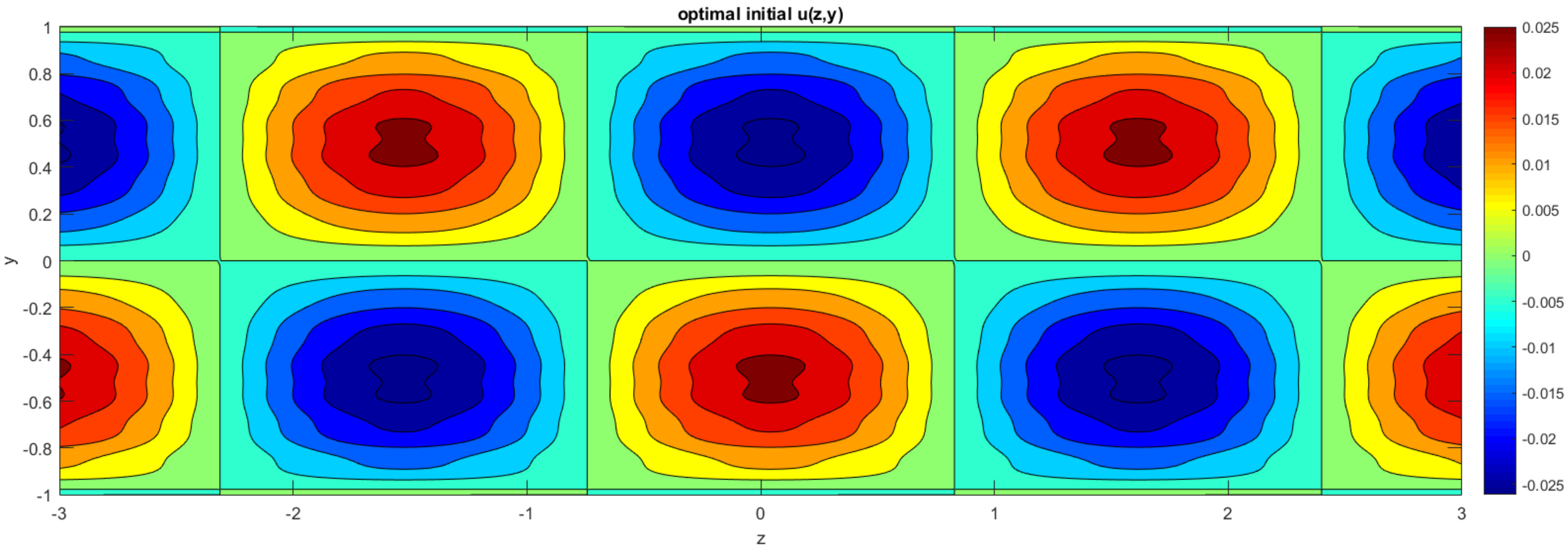


Optimal Initial Condition
$(\alpha = 0, \beta = 2, Re = 1000)$

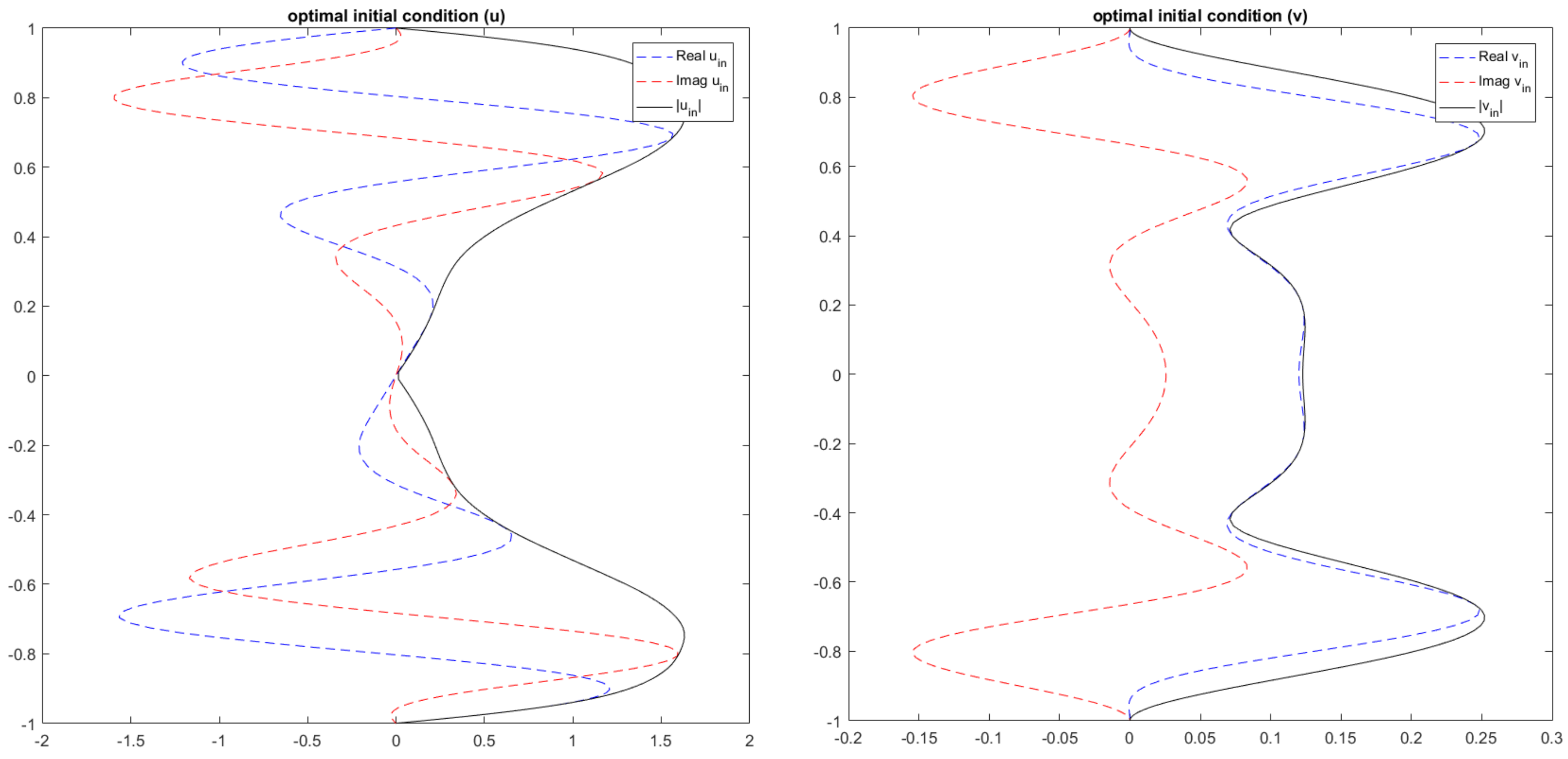




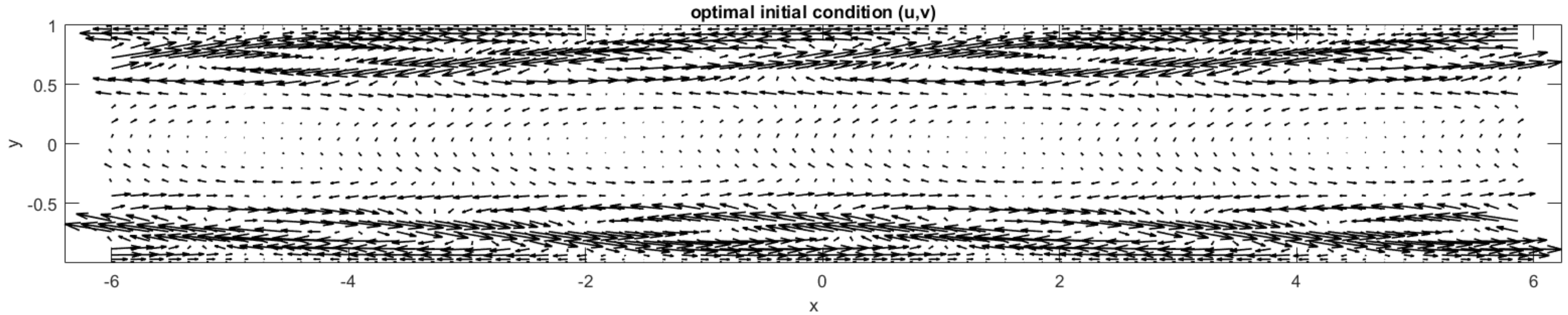
Optimal Initial Condition
$(\alpha = 0, \beta = 2, Re = 1000)$

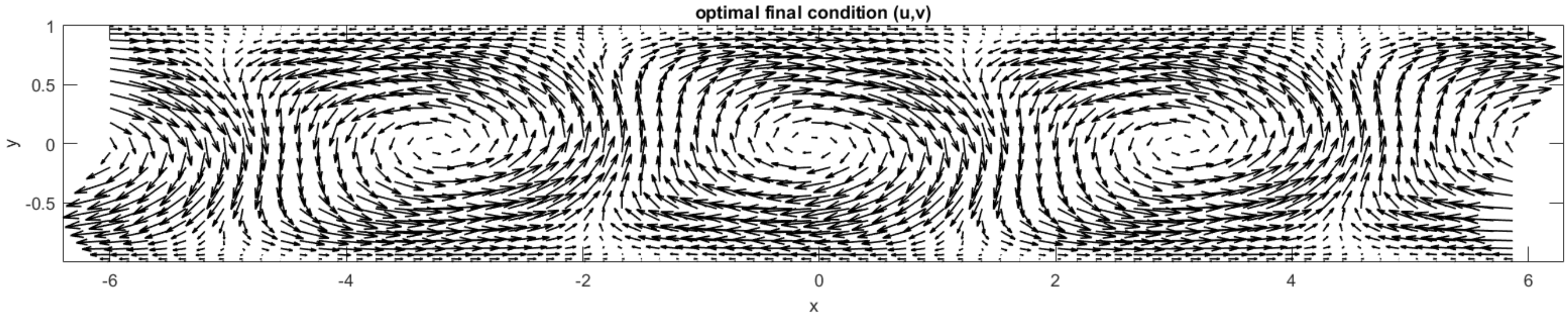


Optimal Initial Condition
$(\alpha = 1, \beta = 0, Re = 1000)$



Optimal Initial Condition
$(\alpha = 1, \beta = 0, Re = 1000)$





Receptivity

Receptivity analysis which is concerned with the general response of a fluid system to external disturbances.
Let us the discretized dynamic system:

$$\frac{d\boldsymbol{q}}{dt} = \boldsymbol{L}\boldsymbol{q} + \boldsymbol{f}.$$

Receptivity is described via a resonance argument, given by the closeness of the external frequencies to any of the eigenvalues of the driven system. However for non-modal system this argument finds out to be inadequate.
The form of the forcing in this analysis will be harmonic and due to linearity of the system the output will respond with the same frequency (we will also consider the case where $\boldsymbol{q_0} = \boldsymbol{0}$).
Analogously to the stability analysis we can obtain the final expression of the resolvent norm as

$$R(\omega) = \max_{\boldsymbol{f}} \frac{\|\boldsymbol{q_{out}}\|_E^2}{\|\boldsymbol{f}\|_E^2} = \|(i\omega\boldsymbol{I} - \boldsymbol{L})^{-1}\|_E^2\,.$$

It is the maximum response due to harmonic forcing, optimized over all forcing.

Receptivity

Applying the eigenvalue decomposition one can rewrite R as:

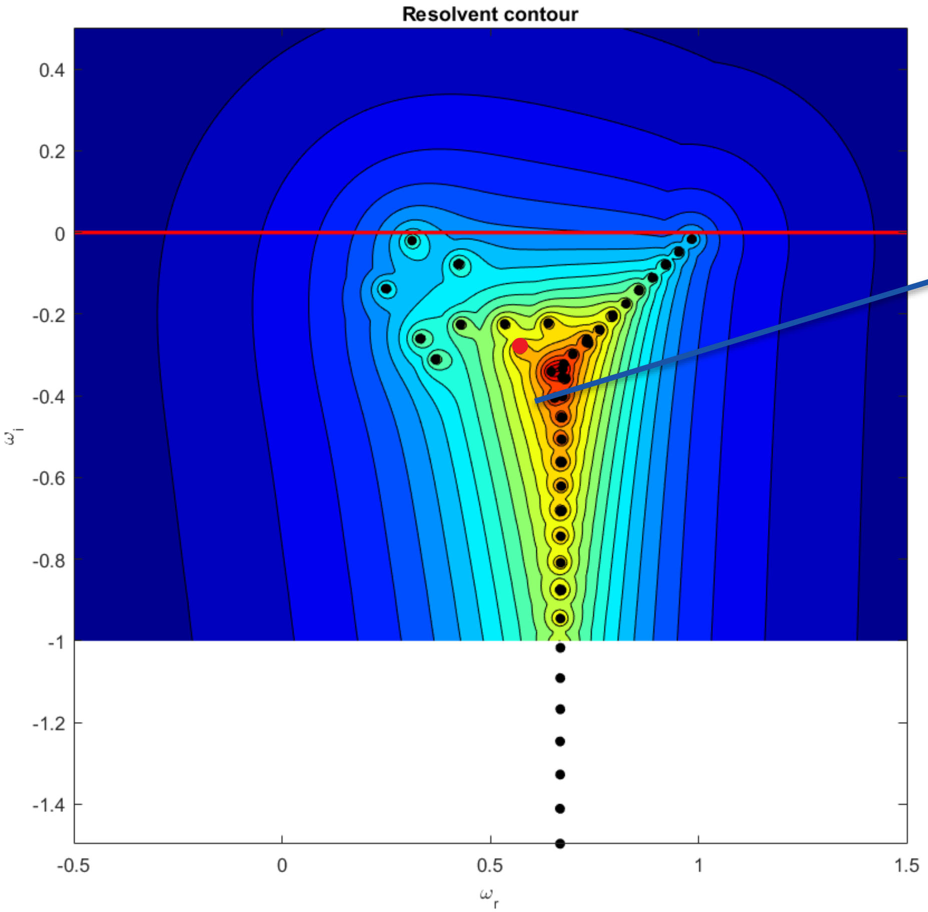
$$R(\omega) = \left\| \boldsymbol{V}^{-\mathbf{1}}(i\omega\boldsymbol{I} - \boldsymbol{\Lambda})^{-1}\boldsymbol{V} \right\|_E^2.$$

Where the inner part containing the diagonal eigenvalues matrix measures the inverse distance of the external forcing frequency with the eigenvalues of our linear system. This is the classical definition of resonance. But it discards any other information regarding the structure of the eigenvectors, which can be non-normal.

The resolvant norm for non-normal system can therfore by very high even though we are not in proximity of an eigenvector.

Receptivity

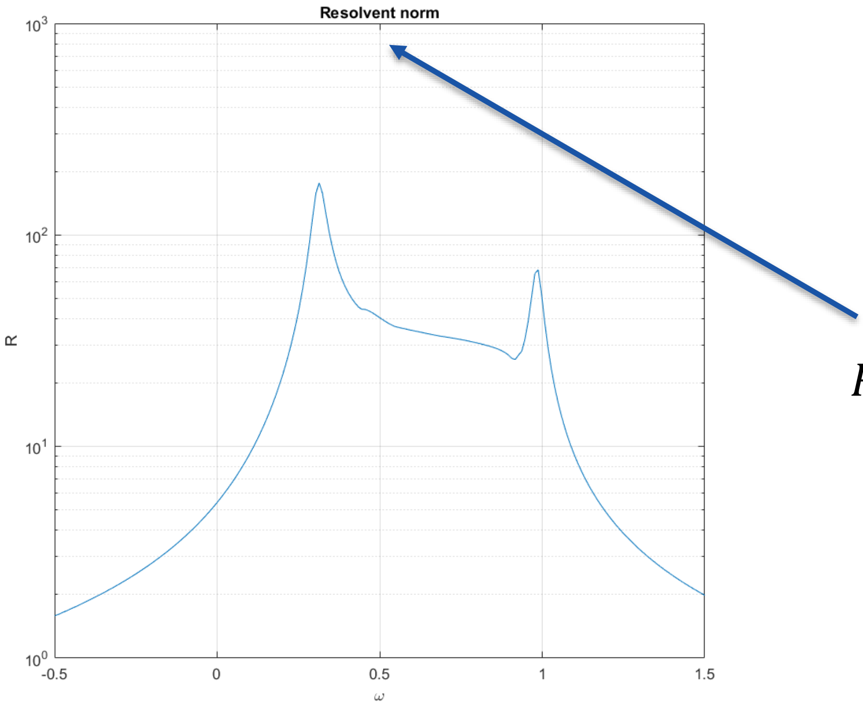
As an exemple the resolvent contour for Plane Poiseuille Flow is showed below



The resolvent norm has the same magnitude in this regon as if it was close to the eigenvalue indicated by the red dot.
This is the region where the eigenvectors are highly non orthogonal.

Optimal Forcing

Analogous to the optimal initial condition problem the optimal forcing identifies the shape of the forcing which produces the largest response in the flow.
Exploiting again the singular value decomposition, the computation of the optimal forcing is straight forward. The computation of the optimal forcing thus amounts to a SVD of the resolvent matrix for a given forcing frequency.



$$\alpha = 1$$
$$\beta = 0$$
$$Re = 2000$$

References

1. Lecture Notes SG2810 Wave Motion and Hydrodinamic Stability.
2. Analysis of fluid systems: stability, receptivity, sensitivity.

   Peter J. Schmid, Luca Brandt

Thank you for the attention