Fluid Mechanics, SG2214, HT2013
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Exercise 1: Tensors and Invariants

Tensor/Index Notation

Scalar (0th order tensor), usually we consider scalar fields function of space and time

$$p = p(x, y, z, t)$$

Vector (1st order tensor), defined by direction and magnitude

$$(\bar{u})_i = u_i \quad \text{If } \bar{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \text{ then } u_2 = v$$

Matrix (2nd order tensor)

$$(\mathbf{A})_{ij} = A_{ij} \quad \text{If } \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ then } A_{23} = a_{23}$$

Kronecker delta (2nd order tensor)

$$\delta_{ij} = (\mathbf{I})_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

To indicate operation among tensor we will use Einstein summation convention (summation over repeated indices)

$$u_i u_i = \sum_{i=1}^{3} u_i u_i \qquad i \text{ is called dummy index (as opposed to free index) and can be renamed}$$

Example: Kinetic energy per unit volume

$$\tfrac{1}{2}\rho|\bar{u}|^2 = \tfrac{1}{2}\rho(u^2 + v^2 + w^2) = \tfrac{1}{2}\rho u_i u_i$$

Matrix/Tensor operations

$$(\bar{a} \cdot \bar{b}) = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_i b_i = \delta_{ij} a_i b_j = a_j b_j \quad \text{(scalar, inner product)}$$

$$(\bar{a}\,\bar{b})_{ij} = (\bar{a} \otimes \bar{b})_{ij} = a_i b_j \quad \text{(diadic, tensor product)}$$

$$(\mathbf{A}\bar{b})_i = A_{ij} b_j \quad \text{(matrix-vector multiplication, inner product)}$$

$$(\mathbf{AB})_{ij} = (\mathbf{A} \cdot \mathbf{B})_{ij} = A_{ik} B_{kj} \quad \text{(matrix multiplication, inner product)}$$

$$(\mathbf{AB})_{ijkl} = (\mathbf{A} \otimes \mathbf{B})_{ijkl} = A_{ij} B_{kl} \quad \text{(diadic, tensor product)}$$

$$(\mathbf{A} : \mathbf{B}) = A_{ij} B_{ij} \quad \text{(double contraction)}$$

$$\mathrm{tr}(\mathbf{A}) = A_{11} + A_{22} + A_{33} = A_{ii} \quad \text{(trace)}$$

$$(\mathbf{A})_{ij} = A_{ij} \iff (\mathbf{A}^T)_{ij} = A_{ji} \quad \text{(transpose)}$$

Permutation symbol

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ in cyclic order. } ijk = 123,\ 231 \text{ or } 312 \\ 0 & \text{if any two indices are equal} \\ -1 & \text{if } ijk \text{ in anticyclic order. } ijk = 321,\ 213 \text{ or } 132 \end{cases}$$

Vector (Cross) product

$$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \bar{e}_1(a_2b_3 - a_3b_2) - \bar{e}_2(a_1b_3 - a_3b_1) + \bar{e}_3(a_1b_2 - a_2b_1) = \varepsilon_{ijk}\bar{e}_i a_j b_k$$

$$(\bar{a} \times \bar{b})_i = \varepsilon_{ijk} a_j b_k$$

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

$$\varepsilon_{ijk}\varepsilon_{ijm} = \{l = j\} = 3\delta_{km} - \delta_{jm}\delta_{kj} = 3\delta_{km} - \delta_{mk} = 2\delta_{km}$$

$$\varepsilon_{ijk}\varepsilon_{ijk} = \{m = k\} = 2 \cdot 3 = 6$$

Example: Rewrite without the cross product:

$$(\bar{a} \times \bar{b}) \cdot (\bar{c} \times \bar{d}) = (\bar{a} \times \bar{b})_i (\bar{c} \times \bar{d})_i = \varepsilon_{ijk} a_j\, b_k \varepsilon_{ilm} c_l d_m = \varepsilon_{ijk}\varepsilon_{ilm}\, a_j\, b_k\, c_l\, d_m =$$
$$(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\, a_j\, b_k\, c_l\, d_m = a_j c_j b_k d_k - a_j d_j b_k c_k = (\bar{a} \cdot \bar{c})(\bar{b} \cdot \bar{d}) - (\bar{a} \cdot \bar{d})(\bar{b} \cdot \bar{c})$$

Tensor Invariants

$$b_{ij},\ b^2_{ij} = b_{ik}b_{kj} \neq (b_{ij})^2,\ b^3_{ij} = b_{ik}b_{kl}b_{lj} \neq (b_{ij})^3$$

Any scalar obtained from a tensor (e.g. b_{ii}, b^2_{ii}, $\ldots$) is invariant, i.e. independent of the coordinate system. The principal invariants are defined by (λ_i are the eigenvalues of $\mathbf{B}$)

$$I_1 = b_{ii} = \mathrm{tr}(\mathbf{B}) = \lambda_1 + \lambda_2 + \lambda_3$$
$$I_2 = \tfrac{1}{2}\left[(b_{ii})^2 - b^2_{ii}\right] = \tfrac{1}{2}\left\{[\mathrm{tr}(\mathbf{B})]^2 - \mathrm{tr}(\mathbf{B}^2)\right\} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$$
$$I_3 = \tfrac{1}{6}(b_{ii})^3 - \tfrac{1}{2}b_{ii}b^2_{jj} + \tfrac{1}{3}b^3_{ii} = \det(\mathbf{B}) = \lambda_1\lambda_2\lambda_3$$

Decomposition of Tensors

$$T_{ij} = T^S_{ij} + T^A_{ij} \quad \textit{symmetric and anti-symmetric parts}$$
$$T^S_{ij} = \tfrac{1}{2}\Big(T_{ij} + T_{ji}\Big) = T^S_{ji} \quad \textit{symmetric}$$
$$T^A_{ij} = \tfrac{1}{2}\Big(T_{ij} - T_{ji}\Big) = -T^A_{ji} \quad \textit{anti-symmetric}$$

The symmetric part of the tensor can be divided further into a trace-less and an isotropic part:

$$T^S_{ij} = \bar{T}_{ij} + \bar{\bar{T}}_{ij}$$
$$\bar{T}_{ij} = T^S_{ij} - \tfrac{1}{3}T_{kk}\delta_{ij} \quad \textit{trace-less}$$
$$\bar{\bar{T}}_{ij} = \tfrac{1}{3}T_{kk}\delta_{ij} \quad \textit{isotropic}$$

This gives:

$$T_{ij} = T^S_{ij} + T^A_{ij} = \bar{T}_{ij} + \bar{\bar{T}}_{ij} + T^A_{ij}$$

In the Navier-Stokes equations we have the tensor $\dfrac{\partial u_i}{\partial x_j}$ (deformation-rate tensor). The anti-symmetric part describes rotation, the isotropic part describes the volume change and the trace-less part describes the deformation of a fluid element.

Operators

$$(\nabla p)_i = \frac{\partial}{\partial x_i} p \quad \text{(gradient, increase of tensor order)}$$

$$\Delta p = \nabla \cdot \nabla p = \nabla^2 p = \frac{\partial^2}{\partial x_i \partial x_i} p \quad \text{(Laplace operator)}$$

$$\nabla \cdot \bar{u} = \frac{\partial}{\partial x_i} u_i \quad \text{(divergence, decrease of tensor order)}$$

$$(\nabla \cdot \mathbf{A})_j = \frac{\partial}{\partial x_i} A_{ij} \quad \text{(divergence of a tensor)}$$

$$(\nabla \bar{u})_{ij} = (\nabla \otimes \bar{u})_{ij} = \frac{\partial}{\partial x_i} u_j \quad \text{(gradient of a vector)}$$

$$(\nabla \times \bar{u})_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} u_k \quad \text{(curl)}$$

Gauss theorem (general)

Gauss theorem (divergence theorem):

$$\oint_S \bar{F} \cdot \bar{n} \, dS = \int_V \nabla \cdot \bar{F} \, dV$$

or with index notation,

$$\oint_S F_i \, n_i \, dS = \int_V \frac{\partial F_i}{\partial x_i} \, dV$$

In general we can write,

$$\oint_S T_{ijk} \, n_l \, dS = \int_V \frac{\partial}{\partial x_l} T_{ijk} \, dV$$

Example: Put $T_{ijk} \, n_l = T_{ij} \, u_l \, n_l$

$$\oint_S T_{ij} \, u_l \, n_l \, dS = \int_V \frac{\partial}{\partial x_l} (u_l \, T_{ij}) \, dV$$

or,

$$\oint_S \mathbf{T} (\bar{u} \cdot \bar{n}) \, dS = \int_V \big((\nabla \cdot \bar{u}) \mathbf{T} + (\bar{u} \cdot \nabla) \mathbf{T} \big) \, dV$$

Identities

- Derive the identity

$$\boxed{\nabla \times (\bar{F} \times \bar{G}) = (\bar{G} \cdot \nabla) \bar{F} + (\nabla \cdot \bar{G}) \bar{F} - (\nabla \cdot \bar{F}) \bar{G} - (\bar{F} \cdot \nabla) \bar{G}}$$

$$\big(\nabla \times (\bar{F} \times \bar{G}) \big)_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} \varepsilon_{klm} F_l \, G_m = \varepsilon_{ijk} \varepsilon_{klm} \frac{\partial}{\partial x_j} F_l \, G_m = \varepsilon_{kij} \varepsilon_{klm} \frac{\partial}{\partial x_j} F_l \, G_m =$$

$$(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\frac{\partial}{\partial x_j}F_l\,G_m = \frac{\partial}{\partial x_j}F_i\,G_j - \frac{\partial}{\partial x_j}F_j\,G_i = \frac{\partial F_i}{\partial x_j}\,G_j + \frac{\partial G_j}{\partial x_j}\,F_i - \frac{\partial F_j}{\partial x_j}\,G_i + \frac{\partial G_i}{\partial x_j}\,F_j =$$

$$G_j\,\frac{\partial F_i}{\partial x_j} + \frac{\partial G_j}{\partial x_j}\,F_i - \frac{\partial F_j}{\partial x_j}\,G_i - F_j\,\frac{\partial G_i}{\partial x_j} = [(\bar{G}\cdot\nabla)\bar{F} + (\nabla\cdot\bar{G})\bar{F} - (\nabla\cdot\bar{F})\bar{G} - (\bar{F}\cdot\nabla)\bar{G}]_i$$

- Show that

$$\boxed{\nabla\cdot(\nabla\times\bar{F}) = 0}$$

$$\nabla\cdot(\nabla\times\bar{F}) = \frac{\partial}{\partial x_i}\varepsilon_{ijk}\frac{\partial}{\partial x_j}F_k = \varepsilon_{ijk}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}F_k$$

Remember that $\varepsilon_{ijk} = -\varepsilon_{jik}$ and that $\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}F_k = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}F_k$ and thus all terms will cancel.

$$\varepsilon_{ijk}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}F_k = 0$$

- Similarly, show that

$$\boxed{\nabla\times(\nabla f) = 0}$$

$\nabla\times(\nabla f) = \varepsilon_{ijk}\frac{\partial^2}{\partial x_j\partial x_k}f = 0$ according to the same argument as above.