Fluid Mechanics, SG2214, HT2013
October 7, 2013

Exercise 10: Axisymmetric flow with vorticity and Rankine vortex

The Rankine vortex

A simple model for a vortex is given by the combination of a rigid-body rotation within a core, and a decay of angular velocity outside. This can be described by

$$u_\theta = \begin{cases} \omega r, & r < a, \\ \dfrac{\omega a^2}{r}, & r > a, \end{cases} \qquad u_r = u_z = 0$$

and is called a Rankine vortex.

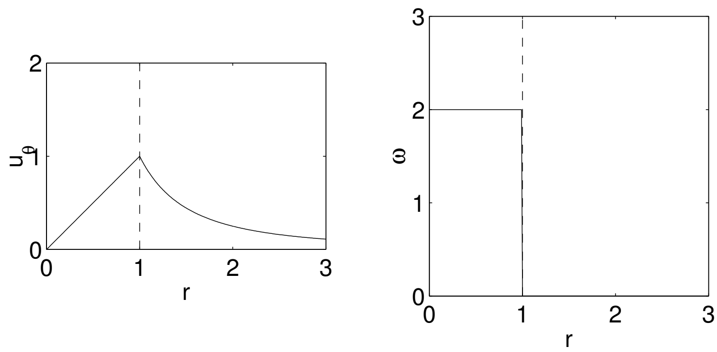

Figure 1: Velocity and vorticity in a Rankine vortex with $\omega = a = 1$.

Example 1: Rankine vortex

Consider the Rankine vortex described above.

a) Find the pressure inside and outside of a Rankine vortex

We use the Euler equations for incompressible flow, i.e. neglecting viscous effects.

$$\text{Euler equations} \quad \begin{cases} \dfrac{D\bar{u}}{Dt} = -\dfrac{1}{\rho}\nabla p + \bar{g} \\ \nabla \cdot \bar{u} = 0 \end{cases}$$

$$\frac{D\bar{u}}{Dt} = \underbrace{\frac{\partial \bar{u}}{\partial t}}_{=0} + (\bar{u} \cdot \nabla)\bar{u}$$

We are working preferably in cylindrical coordinates, use the formulas given in the lecture notes:

$$\bar{u} \cdot \nabla = u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}$$

$$(\bar{u} \cdot \nabla)\bar{u} = \frac{u_\theta}{r}\frac{\partial}{\partial \theta}(u_\theta \bar{e}_\theta) = \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}\bar{e}_\theta + \frac{u_\theta}{r}\underbrace{\frac{\partial \bar{e}_\theta}{\partial \theta}}_{=-\bar{e}_r} u_\theta = \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}\bar{e}_\theta - \frac{u_\theta^2}{r}\bar{e}_r$$

$$\nabla p = \frac{\partial p}{\partial r}\bar{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\bar{e}_\theta + \frac{\partial p}{\partial z}\bar{e}_z$$

Insert this into the Euler equations:

$$\frac{u_\theta}{r}\underbrace{\frac{\partial u_\theta}{\partial \theta}}_{=0}\bar{e}_\theta - \frac{u_\theta^2}{r}\bar{e}_r = -\frac{1}{\rho}\left(\frac{\partial p}{\partial r}\bar{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\bar{e}_\theta + \frac{\partial p}{\partial z}\bar{e}_z\right) - g\bar{e}_z$$

Look at the different components:

$$\bar{e}_r : \quad -\frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\bar{e}_\theta : \quad 0 = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} \Rightarrow p = p(r, z) \quad \text{only.}$$

$$\bar{e}_z : \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

Solve for the pressure when $r < a$:

$$\bar{e}_r : \quad -\omega^2 r = -\frac{1}{\rho}\frac{\partial p}{\partial r} \quad \Rightarrow \quad p = \rho\omega^2\frac{r^2}{2} + f(z)$$

$$\bar{e}_z : \frac{\partial p}{\partial z} = -\rho g \quad \Rightarrow \quad f(z) = -\rho g z + C_1$$

So we obtain for the pressure:

$$\boxed{p(r, z) = \rho\omega^2\frac{r^2}{2} - \rho g z + C_1 \quad \text{for } r < a}$$

Solve for the pressure when $r > a$:

$$\bar{e}_r : \quad -\frac{\omega^2 a^4}{r^3} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \quad \Rightarrow \quad p = -\frac{\rho\omega^2 a^4}{2\,r^2} + f(z)$$

$$\bar{e}_z : \frac{\partial p}{\partial z} = -\rho g \quad \Rightarrow \quad f(z) = -\rho g z + C_2$$

So we obtain for the pressure:

$$\boxed{p(r, z) = -\frac{\rho\omega^2 a^4}{2\,r^2} - \rho g z + C_2 \quad \text{for } r > a}$$

Now determine the difference between the constants C_1 and C_2 by evaluation at $r = a$:

$$\frac{\rho\omega^2 a^2}{2} - \rho g z + C_1 = -\frac{\rho\omega^2 a^2}{2} - \rho g z + C_2 \quad \Rightarrow \quad C_2 - C_1 = \rho\omega^2 a^2$$

b) Determine the pressure difference Δp between $r = 0$ and $r \to \infty$

$$\Delta p = p_\infty - p_0 = -\rho g z + C_2 - (-\rho g z + C_1) = C_2 - C_1 = \rho\omega^2 a^2$$

c) Calculate the shape of a free surface at atmospheric pressure p_0.

Find the difference in z between $r = 0$ and $r \to \infty$

$$\begin{cases} r = 0: \quad p_0 = -\rho g z_0 + C_1 \\ r \to \infty: \quad p_0 = -\rho g z_\infty + C_2 \end{cases} \Rightarrow \quad z_\infty - z_0 = \frac{C_2 - C_1}{\rho g} = \frac{\omega^2 a^2}{g}$$

Determine the shape of the free surface:

$$\begin{cases} p_0 = \frac{\rho\omega^2 r^2}{2} - \rho g z + C_1 \quad r < a \quad z \sim r^2 \Rightarrow z = \frac{\omega^2 r^2}{2g} + \frac{C_1 - p_0}{\rho g} \\ p_0 = -\frac{\rho\omega^2 a^4}{2r^2} - \rho g z + C_2 \quad r > a \quad z \sim \frac{1}{r^2} \Rightarrow z = -\frac{\omega^2 a^4}{2gr^2} + \frac{C_2 - p_0}{\rho g} \end{cases}$$

Set $z = 0$ at $r = 0$. Then $C_1 = p_0$ and we further get

$$z = \frac{\omega^2 r^2}{2g} \quad r < a$$

and

$$z = -\frac{\omega^2 a^4}{2gr^2} + \frac{C_2 - C_1}{\rho g} = -\frac{\omega^2 a^4}{2gr^2} + \frac{\omega^2 a^2}{g} = \frac{\omega^2 a^2}{g}\left(1 - \frac{a^2}{2r^2}\right) \quad r > a$$

So we have

$$z(r) = \begin{cases} \frac{\omega^2 r^2}{2g} & r < a \\ \frac{\omega^2 a^2}{g}\left(1 - \frac{a^2}{2r^2}\right) & r > a\ . \end{cases}$$

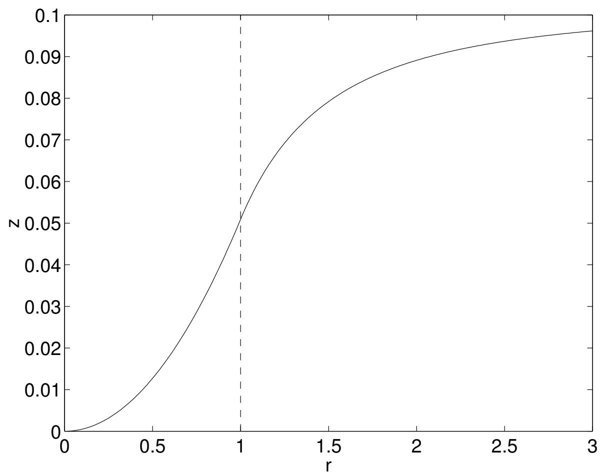

Figure 2: The free surface of a Rankine vortex with $\omega = a = 1$ and $g = 9.82$.

Example 2

Show that the inviscid vorticity equation

$$\frac{D\bar{\omega}}{Dt} = (\bar{\omega} \cdot \nabla)\bar{\mathbf{u}}$$

reduces to the equation

$$\frac{D}{Dt}\left(\frac{\omega}{r}\right) = 0$$

in the case of axisymmetric flow

$$\bar{\mathbf{u}} = u_r(r, z, t)\bar{\mathbf{e}}_r + u_z(r, z, t)\bar{\mathbf{e}}_z\ .$$

The vorticity in an axisymmetric flow

$$\bar{\omega} = \nabla \times \bar{\mathbf{u}} = \underbrace{\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)}_{\omega} \bar{\mathbf{e}}_\theta = \omega \bar{\mathbf{e}}_\theta$$

Study the right hand side of the inviscid vorticity equation

$$(\bar{\omega} \cdot \nabla) = \frac{\omega}{r}\frac{\partial}{\partial \theta} \quad \Rightarrow \quad (\bar{\omega} \cdot \nabla)\bar{\mathbf{u}} = \frac{\omega}{r}\frac{\partial}{\partial \theta}\Big(u_r(r,z,t)\bar{\mathbf{e}}_r + u_z(r,z,t)\bar{\mathbf{e}}_z\Big) =$$

$$\frac{\omega}{r}\underbrace{\frac{\partial u_r}{\partial \theta}}_{=0}\bar{\mathbf{e}}_r + \frac{\omega}{r}u_r\underbrace{\frac{\partial \bar{\mathbf{e}}_r}{\partial \theta}}_{=\bar{\mathbf{e}}_\theta} + \frac{\omega}{r}\underbrace{\frac{\partial u_z}{\partial \theta}}_{=0}\bar{\mathbf{e}}_z + \frac{\omega}{r}u_z\underbrace{\frac{\partial \bar{\mathbf{e}}_z}{\partial \theta}}_{=0} = \frac{\omega}{r}u_r\bar{\mathbf{e}}_\theta$$

The left hand side of the inviscid vorticity equation gives

$$\frac{D\bar{\omega}}{Dt} = \frac{\partial \bar{\omega}}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla)\bar{\omega} = \{\bar{\omega} = \omega\bar{\mathbf{e}}_\theta\} = \left(\frac{\partial \omega}{\partial t} + \left(u_r\frac{\partial}{\partial r} + u_z\frac{\partial}{\partial z}\right)\omega\right)\bar{\mathbf{e}}_\theta$$

This gives that the inviscid vorticity equation now is

$$\frac{\partial \omega}{\partial t} + \left(u_r\frac{\partial}{\partial r} + u_z\frac{\partial}{\partial z}\right)\omega = \frac{\omega}{r}u_r$$

Multiply by $\frac{1}{r}$

$$\frac{\partial}{\partial t}\left(\frac{\omega}{r}\right) + \frac{1}{r}\left(u_r\frac{\partial}{\partial r} + u_z\frac{\partial}{\partial z}\right)\omega - \frac{\omega}{r^2}u_r = 0$$

Notice that

$$-\frac{\omega}{r^2}u_r = \omega u_r\frac{\partial}{\partial r}\frac{1}{r} \quad \text{and that} \quad \omega u_z\frac{\partial}{\partial z}\frac{1}{r} = 0$$

This means we can write

$$\frac{\partial}{\partial t}\left(\frac{\omega}{r}\right) + \frac{1}{r}\left(u_r\frac{\partial}{\partial r} + u_z\frac{\partial}{\partial z}\right)\omega + \omega\left(u_r\frac{\partial}{\partial r} + u_z\frac{\partial}{\partial z}\right)\frac{1}{r} = 0$$

And thus we have

$$\frac{\partial}{\partial t}\left(\frac{\omega}{r}\right) + \frac{1}{r}\left(u_r\frac{\partial}{\partial r} + u_z\frac{\partial}{\partial z}\right)\frac{\omega}{r} = \frac{D}{Dt}\left(\frac{\omega}{r}\right) = 0$$

Example 3: Inviscid and Irrotational Vortices

Consider a circular flow with $\bar{u} = u_\theta(r)\bar{e}_\theta$. Which vortices are inviscid and which vortices are irrotational?

The Navier–Stokes equation for u_θ

$$\frac{\partial u_\theta}{\partial t} = -\frac{1}{\rho\, r}\frac{\partial p}{\partial r} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) - \frac{u_\theta}{r^2}\right)$$

For inviscid flow we require,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) - \frac{u_\theta}{r^2} = 0$$

Make the ansatz $u_\theta = r^n$,

$$0 = \frac{1}{r}\frac{\partial}{\partial r}(rnr^{n-1}) - r^{n-2} = \frac{1}{r}n^2r^{n-1} - r^{n-2} \quad \rightarrow \quad n^2 - 1 = 0 \quad \Rightarrow n = \pm 1$$

We get the inviscid flow,

$$u_\theta(r) = \underbrace{Ar}_{\text{solid body rotation}} + \underbrace{\frac{B}{r}}_{\text{irrotational}}$$

The vorticity is,

$$\bar{\omega} = \nabla \times \bar{u} = \{A.32\} = \frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\,\bar{e}_z$$

$$\bar{\omega} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial r}(ru_\theta) = 0 \quad \Rightarrow \quad u_\theta = \frac{C}{r}$$

Conclusion:

Irrotational $\Rightarrow$ inviscid
Inviscid $\not\Rightarrow$ irrotational