Fluid Mechanics, SG2214, HT2013
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Exercise 11: Vorticity, Bernoulli and Stream Function

Example 1: Solid-Body Rotation

Consider the flow in an uniformly rotating bucket with velocity

$$\mathbf{u} = (\omega r, 0, 0)$$

a) Use Bernoulli equation to determine the free surface: What is wrong?

Bernoulli:

$$\frac{p}{\rho} + \frac{1}{2}|\mathbf{u}|^2 + gz = C \quad \text{along streamlines}$$

This gives the surface of constant pressure

$$z = \frac{C - p_0}{\rho g} - \frac{\omega^2 r^2}{2g}$$

The free surface is highest in the center of the bucket, something is wrong. Bernoulli theorem is valid along a streamline in a steady ideal fluid. If the flow had been irrotational, it would have been valid everywhere. But now $\nabla \times \bar{\mathbf{u}} = (0, 0, 2\omega)$.

b) Use Euler equation to determine the free surface:

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{g}$$

this gives

$$\mathbf{e}_r : \quad -\frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \quad \Rightarrow \quad \frac{\partial p}{\partial r} = \rho\omega^2 r \quad \Rightarrow \quad p = \frac{1}{2}\rho\omega^2 r^2 + f(z)$$

$$\mathbf{e}_\theta : \quad 0 = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \quad \Rightarrow \quad \frac{\partial p}{\partial \theta} = 0$$

$$\mathbf{e}_z : 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \quad \Rightarrow \quad \frac{\partial f}{\partial z} = -\rho g \quad \Rightarrow \quad f = \rho g z + p_0$$

Thus

$$p(r, z) = \frac{1}{2}\rho\omega^2 r^2 - \rho g z + p_0$$

This gives the free surface where $p = p_0$

$$z = \frac{\omega^2 r^2}{2g}$$

Example 2: Flow over a Hill

A hill with the height h has the shape of a half circular cylinder as shown in Figure 1. Far from the hill the wind U_∞ is blowing parallel to the ground in the x-direction and the atmospheric pressure at the ground is p_0.

a) Assume potential flow and show that the stream function in cylindrical coordinates is of the form

$$\psi = f(r)\sin\theta,$$

where $f(r)$ is an arbitrary function. Calculate the velocity field above the hill.

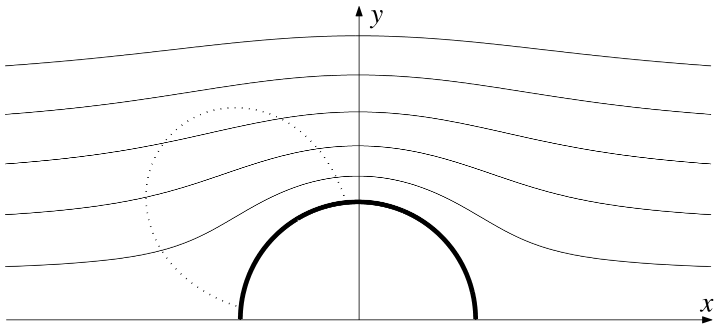

Figure 1: Streamlines above a hill with $h = 100m$ and $U_\infty = 5m/s$. A paraglider pilot with a sink of $1m/s$ will find lift in the area within the dotted line, while soaring along the hill.

The stream function satisfies continuity:

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \psi}{\partial r}$$

The flow is irrotational:

$$\overline{\omega} = \nabla \times \bar{u} = 0 \quad \Rightarrow \quad \frac{\partial \psi}{\partial r} + r\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial^2 \psi}{\partial \theta^2} = 0 \quad (1)$$

Introduce the ansatz $\psi = f(r)\sin\theta$ into equation (1):

$$f'\sin\theta + rf''\sin\theta - \frac{1}{r}f\sin\theta = 0 \quad \Rightarrow$$

$$f' + rf'' - \frac{1}{r}f = 0$$

Make the ansatz $f = r^n$:

$$nr^{n-1} + rn(n-1)r^{n-2} - \frac{1}{r}r^n = 0 \quad \Rightarrow$$

$$n + n^2 - n - 1 = 0 \quad \Rightarrow \quad n = \pm 1$$

So we have

$$\psi = \left(Ar + \frac{B}{r}\right)\sin\theta$$

We need two boundary conditions.
1. Free stream:

$$Ar\sin\theta = U_\infty r\sin\theta \quad \Rightarrow \quad A = U_\infty$$

2. Streamline on the hill surface:

$$U_\infty h + \frac{B}{h} = 0 \quad \Rightarrow \quad B = -U_\infty h^2$$

So we have:

$$\psi = U_\infty\left(r - \frac{h^2}{r}\right)\sin\theta$$

Now we can calculate the velocity field above the hill:

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U_\infty\left(1 - \frac{h^2}{r^2}\right)\cos\theta$$

$$u_\theta = -\frac{\partial \psi}{\partial r} = -U_\infty\left(1 + \frac{h^2}{r^2}\right)\sin\theta$$

b) Derive an equation for the curve with constant vertical wind velocity V.

Constant vertical wind velocity is described by:

$$V = u_r \sin\theta + u_\theta \cos\theta = U_\infty \left(1 - \frac{h^2}{r^2}\right)\cos\theta \sin\theta - U_\infty \left(1 + \frac{h^2}{r^2}\right)\sin\theta\cos\theta = -2U_\infty \frac{h^2}{r^2}\sin\theta\cos\theta \quad \Rightarrow$$

$$\underline{r = h\sqrt{-2\frac{U_\infty}{V}\sin\theta\cos\theta}}$$

c) Assume that the density ρ and the gravitational acceleration g is constant. Calculate the atmospheric pressure at the top of the hill.

Use the Bernoulli equation (valid everywhere) with free stream pressure p_0 at the ground:

$$p_o + \frac{1}{2}\rho U_\infty^2 = p + \frac{1}{2}\rho(2U_\infty)^2 + \rho g h \quad \Rightarrow p = p_o - \frac{3}{2}\rho U_\infty^2 - \rho g h$$

Stokes stream function

Consider a 2D incompressible flow

$$\nabla \cdot \bar{u} = 0 \quad \text{or} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Define the stream function Ψ such that:

$$u = \frac{\partial \Psi}{\partial y} \qquad v = -\frac{\partial \Psi}{\partial x}$$

This means

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial^2 \Psi}{\partial y \partial x} = 0,$$

so continuity is always fulfilled. Now we can write

$$\bar{u} = \nabla \times \Psi \bar{\mathbf{e}}_z = \begin{vmatrix} \bar{\mathbf{e}}_x & \bar{\mathbf{e}}_y & \bar{\mathbf{e}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \Psi \end{vmatrix} = \left(\frac{\partial \Psi}{\partial y}, -\frac{\partial \Psi}{\partial x}, 0\right)$$

And

$$\bar{\omega} = \nabla \times \bar{u} = \begin{vmatrix} \bar{\mathbf{e}}_x & \bar{\mathbf{e}}_y & \bar{\mathbf{e}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \Psi}{\partial y} & -\frac{\partial \Psi}{\partial x} & 0 \end{vmatrix} = \left(0, 0, -\frac{\partial \Psi}{\partial x^2} - \frac{\partial^2 \Psi}{\partial y^2}\right) = -\nabla^2 \Psi \bar{\mathbf{e}}_z$$

For irrotational flow

$$\Delta \Psi = \nabla^2 \Psi = 0$$

In spherical coordinates for axisymmetrical flow, define Ψ

$$u_r = \frac{1}{r^2 \sin\theta}\frac{\partial \Psi}{\partial \theta} \qquad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial \Psi}{\partial r}$$

Incompressibility is still valid

$$\nabla \cdot \bar{u} = 0$$

Velocity

$$\bar{u} = \nabla \times \frac{\Psi}{r\sin\theta}\bar{\mathbf{e}}_\theta$$

Vorticity

$$\bar{\omega} = -\frac{1}{r\sin\theta}\left[\frac{\partial^2 \Psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial \Psi}{\partial \theta}\right)\right]$$

Irrotational

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial \Psi}{\partial \theta}\right) = 0$$

Example 3: Flow around a Sphere

Consider a sphere with $\Psi = 0$ at $r = a$, compute the irrotational velocity distribution when the velocity of the freestream at infinity is U:

$$r \to \infty \quad \Psi \to \frac{1}{2} U r^2 \sin^2\theta$$
$$\Rightarrow u_r \to U\cos\theta \quad u_\theta \to -U\sin\theta \ .$$

Make the ansatz:

$$\Psi = f(r)\sin^2\theta$$

For an irrotational flow we get

$$f'' - \frac{2}{r^2} f = 0 \quad \Rightarrow \quad f = Ar^2 + \frac{B}{r}$$

From the boundary conditions at infinity we get

$$A = \frac{1}{2} U$$

On the surface of the sphere ($r = a$)

$$\frac{1}{2} U a^2 + \frac{B}{a} = 0 \quad \Rightarrow \quad B = -\frac{1}{2} U a^3$$

This gives

$$\Psi = \frac{1}{2} U \left(r^2 - \frac{a^3}{r} \right) \sin^2\theta$$

The slip velocity on the sphere is

$$u_\theta = -\frac{3}{2} U \sin\theta$$

The radial velocity is of course $u_r = 0$ on the surface.