Fluid Mechanics, SG2214, HT2013
September 16, 2013

Exercise 5: Exact Solutions to the Navier-Stokes Equations II

Example 1: Stokes Second Problem

Consider the oscillating Rayleigh-Stokes flow (or Stokes second problem) as in figure 1.

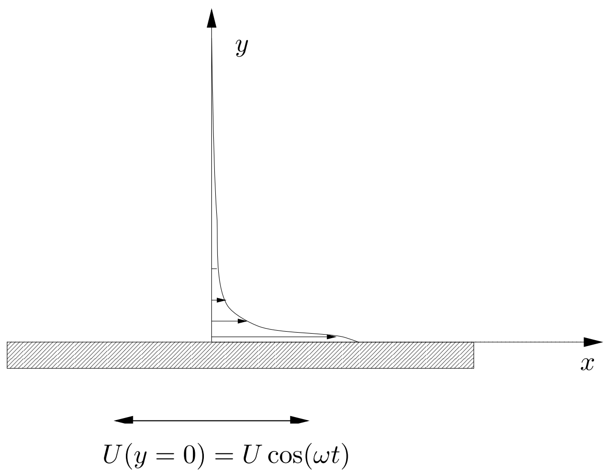

Figure 1: Coordinate system for the Rayleigh-Stokes flow

a) Show that the velocity field $\bar{u} = [u(y,t), 0, 0]$ satisfies the equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

Consider the Navier–Stokes equation in the x direction:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right).$$

With the current velocity field only terms with y derivatives will remain since there can be no change in the other directions. Furthermore, the streamwise pressure gradient has to be zero since the streamwise velocity far from the wall is constant, namely zero.

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

b) Show that the velocity field is

$$u(y,t) = Ue^{-ky}\cos(ky - \omega t), \quad \text{where} \quad k = \sqrt{\frac{\omega}{2\nu}}$$

Make the ansatz: $u = \Re\Big[f(y)e^{i\omega t}\Big] = f(y)\cos(\omega t)$.
Insert into the equation:

$$i\omega f(y)e^{i\omega t} = \nu f''(y)e^{i\omega t}$$

$$f''(y) - \frac{i\omega}{\nu} f(y) = 0 \quad \text{with } f(y) = e^{\lambda y} \quad \text{gives}$$

$$\lambda^2 - \frac{i\omega}{\nu} = 0 \quad \Rightarrow \quad \lambda = \pm\sqrt{\frac{i\omega}{\nu}} = \pm\sqrt{\frac{\omega}{\nu}}\left(\frac{1+i}{\sqrt{2}}\right)$$

Introduce $k = \sqrt{\frac{\omega}{2\nu}}$,

$$f(y) = Ae^{yk(1+i)} + Be^{-yk(1+i)}, \quad f(y) \to 0 \quad \text{as} \quad y \to \infty \to A = 0$$

We have

$$u = \Re\left[Be^{-yk(1+i)}e^{iwt}\right] = \Re\left[Be^{-ky}e^{i(\omega t - ky)}\right] = Be^{-ky}\cos(\omega t - ky) = Be^{-ky}\cos(ky - \omega t)$$

Boundary condition at $y = 0$

$$\text{At } y = 0 \text{ we have } u = U\cos(\omega t) \Rightarrow B = U$$

So we have (see figure 2)

$$\boxed{u(y,t) = Ue^{-ky}\cos(ky - \omega t), \quad \text{where} \quad k = \sqrt{\frac{\omega}{2\nu}}}$$

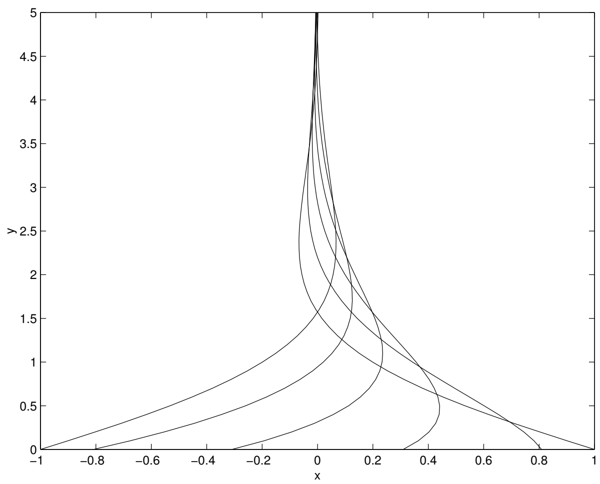

Figure 2: Flow over an oscillating wall, exercise 1b).

c) How thick is the boundary-layer thickness?

If we define the thickness δ of the oscillating layer as the position where $u/U = 0.01$ we get that

$$e^{-k\delta} = 0.01 \Rightarrow k\delta \approx 4.6 \Rightarrow \delta \approx 4.6\sqrt{\frac{2\nu}{\omega}} \Rightarrow \delta \approx 6.5\sqrt{\frac{\nu}{\omega}}$$

d) Consider instead the oscillating flow $U_\infty = U\cos(\omega t)$ over a stationary wall. This will simply result in a change of the reference frame to one following the plate instead. If we consider the solution to the previous problem and look at it in this new frame of reference we get

$$u(y,t) = U\cos(\omega t) - Ue^{-ky}\cos(ky - \omega t), \quad \text{where} \quad k = \sqrt{\frac{\omega}{2\nu}}\ .$$

The solution is shown in figure 3.

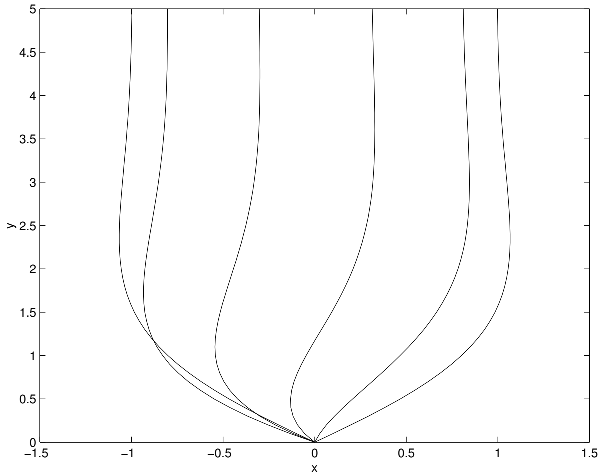

Figure 3: Oscillating flow over a stationary wall, exercise 1d).

Example 2

Consider a long hollow cylinder with inner radius r_1 and a concentric rod with radius r_0 inside it. The rod is moving axially with velocity U_0.

a) Find the velocity field of a viscous fluid occupying the space between the rod and the cylinder.

Assumptions:
• Steady flow:

$$\frac{\partial}{\partial t} = 0$$

• Parallel flow and symmetry:

$$\mathbf{u} = u_z(r)\mathbf{e}_z, \quad \frac{\partial}{\partial z} = 0, \quad \frac{\partial}{\partial \theta} = 0$$

• No axial pressure gradient:

$$\frac{\partial p}{\partial z} = 0$$

We can directly see that $u_r = u_\theta = 0$ satisfy the two first components of the Navier-Stokes equations (i.e. the radial and azimuthal directions). The streamwise momentum equation reduces to

$$(\mathbf{u} \cdot \nabla)u_z = \nu \nabla^2 u_z$$

where

$$(\mathbf{u} \cdot \nabla)u_z = u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} = 0$$

$$\nabla^2 u_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) .$$

We obtain

$$\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = 0 .$$

Integrate twice

$$r\frac{\partial u_z}{\partial r} = A \quad \Rightarrow \quad u_z = A \ln r + B$$

Boundary conditions

$$\begin{cases} u_z(r_0) &= U_0 &= A\ln r_0 + B \\ u_z(r_1) &= 0 &= A \ln r_1 + B \quad \Rightarrow \quad B = -A\ln r_1 \end{cases} \quad \Rightarrow$$

$$U_0 = A(\ln r_0 - \ln r_1) \quad \Rightarrow \quad A = \frac{U_0}{\ln \frac{r_0}{r_1}}$$

$$u_z(r) = \frac{U_0}{\ln \frac{r_0}{r_1}} \ln r - \frac{U_0}{\ln \frac{r_0}{r_1}} \ln r_1 = U_0 \frac{\ln \frac{r}{r_1}}{\ln \frac{r_0}{r_1}}$$

b) With what force does one have to pull a rod with length L? Neglect end effects.

Shear stress

$$\tau_{rz} = \mu \frac{\partial u_z}{\partial r} = \frac{\mu U_0}{r \ln \frac{r_0}{r_1}}$$

Force

$$F = 2\pi r_0 L \tau_{rz}(r_0) = \frac{2\pi L \mu U_0}{\ln \frac{r_0}{r_1}}$$

Example 3: Asymptotic Suction Boundary Layer

Calculate the asymptotic suction boundary layer, where the boundary layer over a flat plate is kept parallel by a steady suction V_0 through the plate.

Assumptions:
Two-dimensional flow:

$$\frac{\partial}{\partial z} = 0, \quad w = 0$$

Parallel, fully-developed flow:

$$\frac{\partial}{\partial x} = 0$$

Steady flow:

$$\frac{\partial}{\partial t} = 0$$

Momentum equations:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Normal momentum equation gives

$$\frac{\partial p}{\partial y} = 0$$

Boundary conditions:

$$y = 0: \quad u = 0, \quad v = -V_0$$

$$y \to \infty: \quad u \to U_\infty$$

Continuity gives

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \Rightarrow \quad v = -V_0$$

Streamwise momentum equation at $y \to \infty$

$$-V_0 \frac{\partial U_\infty}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 U_\infty}{\partial y^2}$$

$$\Rightarrow \quad \frac{\partial p}{\partial x} = 0$$

Resulting streamwise momentum equation

$$-V_0 \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \quad \Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = -\frac{V_0}{\nu} \frac{\partial u}{\partial y}$$

Characteristic equation

$$\lambda^2 = -\frac{V_0}{\nu}\lambda \quad \Rightarrow \quad \lambda_1 = 0, \lambda_2 = -\frac{V_0}{\nu}$$

$$u(y) = A + Be^{-V_0 y/\nu}$$

With the boundary conditions at $y = 0$ and $y = \infty$ we get

$$\boxed{u(y) = U_\infty \left(1 - e^{-V_0 y/\nu}\right)} \, .$$