Exercise 6: Boundary layers: Similarity and Wake Flow

Example 1: Converging Channel

Consider the high Reynolds-number flow in a converging channel. Compute the boundary layer over the surface at \( y = 0 \).

Assume the free stream:

\[
U(x) = -\frac{Q}{x},
\]

where \( Q \) is the flux. The boundary layer equations read:

\[
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{Q}{x^3} + \nu \frac{\partial^2 u}{\partial y^2} \tag{1}
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}
\]

Seek a similarity solution where the dimensionless velocity \( u/U \) only depend on the dimensionless wall distance

\[
\eta = \frac{y}{\delta} = \frac{y}{x \sqrt{\frac{\nu}{Q}}}
\]

The stream function satisfies (2):

\[
u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}
\]

Make the stream function dimensionless:

\[
f(\eta) = \frac{\psi}{U \delta}
\]

Determine the terms in equation (1):

\[
\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \left( -\frac{Q}{x} f' \right) = -\frac{Q}{x} \frac{\partial f'}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{Q}{x^2} f' = -\frac{Q}{x} f'' \left( -\frac{\eta}{x} \right) + \frac{Q}{x^2} f' = \frac{Q}{x^2} \left( \eta f'' + f' \right)
\]

\[
\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left( -\frac{Q}{x} f' \right) = -\frac{Q}{x} \frac{\partial f'}{\partial \eta} \frac{\partial \eta}{\partial y} = -\frac{Q}{x} f'' \left( -\frac{\eta}{x} \right) = -\frac{Q}{x^2} f''
\]

\[
\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \left( -\frac{Q}{x} f'' \left( -\frac{\eta}{x} \right) \right) = -\frac{Q}{x^2} \frac{\partial f''}{\partial \eta} \frac{\partial \eta}{\partial y} = -\frac{Q}{x^2} \frac{\partial f''}{\partial \eta} \frac{\partial \eta}{\partial y} = -\frac{Q^2}{x^3 \nu} f''
\]

Insert into equation (1):

\[
-\frac{Q}{x} f' \frac{Q}{x^2} \left( \eta f'' + f' \right) + \sqrt{\frac{\nu}{Q}} \frac{Q}{x^2} \sqrt{\frac{Q}{\nu}} f'' = -\frac{Q^2}{x^3} - \nu \frac{Q^2}{x^3 \nu} f''
\]
Divide by $Q^2/x^3$:

$$-\eta f' f'' - f'^2 + \eta f' f''' = -1 - f''' \Rightarrow$$

$$f''' - f'^2 + 1 = 0$$

Boundary conditions:

$$\begin{cases} u(0) = 0 \\ u(\infty) = 1 \end{cases} \Rightarrow \begin{cases} f'(0) = 0 \\ f'(\infty) = 1 \end{cases}$$

Substitute $F(\eta) = f'(\eta)$:

$$\begin{cases} F'' - F^2 + 1 = 0 \\ F(0) = 0 \\ F(\infty) = 1 \end{cases}$$

Solution:

$$F = 3 \tanh^2 \left( \frac{\eta}{\sqrt{2}} + \sqrt{\frac{2}{3}} \right) - 2$$

**Example 2: Wake Flow**

Consider the flow downstream of a 2D streamlined body at high Reynolds number. Study the thin wake downstream of the body where variations in $y$ are much more rapid than variations in the downstream direction. Also assume that the wake is so small that we can write $u = U + u_1$ where $u_1$ is negative and describes the wake. The length scale in $x$ is $L$ and in $y$ it is $\delta$ with $\delta << L$.

![Figure 1: Coordinate system for wake problem](image)

a) Find the governing (linear) equations:

We start with the boundary layer equations since $\delta << L$,

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}$$

In the wake we have no pressure gradient,

$$\frac{\partial p}{\partial x} = 0.$$
From the continuity equation we get,
\[
\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{\partial}{\partial x} \left( U + u_1 \right) = -\frac{\partial u_1}{\partial x} \quad \Rightarrow \quad v \sim \frac{\delta}{L} u_1 \ll U
\]

Inserting \( u = U + u_1 \) in the boundary layer equations gives,
\[
U \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_1}{\partial x} + v \frac{\partial u_1}{\partial y} = 0 + \nu \frac{\partial^2 u_1}{\partial y^2}
\]

Neglect the quadratic terms,
\[
u_1 \frac{\partial u_1}{\partial x} \sim \frac{u_1^2}{L} \quad \text{and} \quad v \frac{\partial u_1}{\partial y} \sim \frac{\delta}{L} u_1 \frac{1}{\delta} u_1 \sim \frac{u_1^2}{L}
\]
\[
\Rightarrow \quad \frac{U}{\nu} \frac{\partial u_1}{\partial x} = \nu \frac{\partial^2 u_1}{\partial y^2}
\]

b) Show that \( \int_{-\infty}^{\infty} u_1 \, dy = \text{constant} \):

Consider the relation,
\[
Q_1 = \int_{-\infty}^{\infty} u_1 \, dy
\]
which gives the linear contribution from \( u_1 \) to the momentum flux.
\[
\frac{d}{dx} Q_1 = \frac{d}{dx} \int_{-\infty}^{\infty} u_1 \, dy = \int_{-\infty}^{\infty} \frac{\partial u_1}{\partial x} \, dy = \{\text{From the governing equation}\} = \int_{-\infty}^{\infty} \frac{\nu}{U} \frac{\partial^2 u_1}{\partial y^2} \, dy = \frac{\nu}{U} \left[ \frac{\partial u_1}{\partial y} \right]_{-\infty}^{\infty} = 0
\]
This means that \( Q_1 \) is constant.

c) Find a similarity solution for \( u_1 \):

Seek a solution on the form,
\[
u_1(x, y) = F(x) f(\eta) \quad \text{where} \quad \eta = \frac{y}{g(x)}
\]
This gives,
\[
Q_1 = \int_{-\infty}^{\infty} F(x) f(\eta) \, d\eta = F(x) g(x) \int_{-\infty}^{\infty} f(\eta) \, d\eta
\]
\( Q_1 = \text{constant} \) requires,
\[
F(x) = 1/g(x) \quad \text{and} \quad \int_{-\infty}^{\infty} f(\eta) \, d\eta = Q_1 = \text{constant}
\]
This means,
\[
u_1(x, y) = \frac{1}{g(x)} f(\eta) = \frac{1}{g(x)} f \left( \frac{y}{g(x)} \right)
\]
Insert this into the equation of motion,
\[
-U \frac{g'(x)}{g(x)^2} \left( \eta f'(\eta) + f(\eta) \right) = \nu \frac{1}{g(x)^2} f''(\eta) \quad \Rightarrow \quad \frac{U}{\nu} \frac{g'(x)}{g(x)^2} \left( \eta f'(\eta) + f(\eta) \right) = C \quad \Rightarrow \quad g(x) = \left( 2C \nu x U \right)^{1/2}
\]
This gives the equation for $f$,

$$f''(\eta) + C \eta f'(\eta) + Cf(\eta) = \frac{d}{d\eta} \left( f'(\eta) + C \eta f(\eta) + D \right) = 0$$

From the symmetry condition $\frac{\partial}{\partial y} u(x, 0) = 0$ we can determine that $f'(0) = -D = 0$. This gives,

$$f'(\eta) + C \eta f(\eta) = 0$$

Integrating this we get,

$$f(\eta) = ae^{-\frac{1}{2}C\eta^2}$$

We can determine the constant $a$ from the condition that $Q_1$ is constant,

$$\int_{-\infty}^{\infty} f(\eta) d\eta = Q_1 \quad \Rightarrow \quad f(\eta) = \sqrt{\frac{C}{2\pi}} Q_1 e^{-\frac{1}{2}C\eta^2}$$

This now gives $u_1$:

$$u_1(x, y) = \sqrt{\frac{C}{2\pi}} Q_1 \frac{\nu x}{U} e^{-\frac{1}{2}C\nu^2 \frac{y^2}{U^2}}$$

\begin{equation}
\begin{aligned}
\{ g(x) = \left( \frac{2C \nu x}{U} \right)^{1/2} \} = \frac{Q_1}{2\sqrt{\pi}} \left( \frac{U}{\nu x} \right)^{1/2} e^{-\frac{U y^2}{4\nu x}}
\end{aligned}
\end{equation}

d) Relate $u_1$ to the drag $F_D$:

The drag is

$$F_D = -\rho \int_{-\infty}^{\infty} u_1(U + u_1) dy$$

Remember that $u_1 << U$ and neglect the $u_1^2$ term,

$$F_D = -\rho \int_{-\infty}^{\infty} U u_1 dy = -\rho U Q_1$$

This means that

$$Q_1 = \frac{F_D}{\rho U} \quad \Rightarrow \quad u_1(x, y) = -\frac{F_D}{2\mu \sqrt{\pi}} \left( \frac{\nu}{U x} \right)^{1/2} e^{-\frac{U y^2}{4\nu x}}$$
Example 3: Reynolds Number

Give an order of magnitude estimate of the Reynolds number for:

i. Flow past the wing of a jumbo jet at 150 m/s (≈ Mach 0.5)

ii. A wing profile in salt water with $L = 2$ cm and $U = 5$ cm/s

iii. A thick layer of golden syrup draining of a spoon.

iv. A spermatozoan with tail length of $10^{-3}$ cm swimming at $10^{-1}$ cm/s in water.

Estimate the boundary layer thickness in case (i).

<table>
<thead>
<tr>
<th>Fluid</th>
<th>$\nu$ cm$^2$/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water</td>
<td>0.01</td>
</tr>
<tr>
<td>Air</td>
<td>0.15</td>
</tr>
<tr>
<td>Syrup</td>
<td>1200</td>
</tr>
</tbody>
</table>

i. Flow past the wing of a jumbo jet at 150 m/s (≈ Mach 0.5)

\[
U = 150 \text{ m/s}, \nu = 1.5 \times 10^{-5} \text{ m}^2/\text{s}, \quad L = 4 \text{ m} \Rightarrow Re = \frac{UL}{\nu} \approx 4 \times 10^7
\]

ii. A wing profile in salt water with $L = 2$ cm and $U = 5$ cm/s

\[
U = 0.05 \text{ m/s}, \nu = 10^{-6} \text{ m}^2/\text{s}, \quad L = 0.02 \text{ m} \Rightarrow Re = \frac{UL}{\nu} \approx 10^3
\]

iii. A thick layer of golden syrup draining of a spoon.

\[
U = 0.04 \text{ m/s}, \nu = 0.12 \text{ m}^2/\text{s}, \quad L = 0.01 \text{ m} \Rightarrow Re = \frac{UL}{\nu} \approx 0.003
\]

iv. A spermatozoan with tail length of $10^{-3}$ cm swimming at $10^{-2}$ cm/s in water.

\[
U = 10^{-4} \text{ m/s}, \nu = 10^{-6} \text{ m}^2/\text{s}, \quad L = 10^{-5} \text{ m} \Rightarrow Re = \frac{UL}{\nu} \approx 10^{-3}
\]

The boundary layer thickness in (i) is $O(1 \text{ mm})$

\[
\frac{\delta}{L} = O\left(\frac{1}{\sqrt{Re}}\right)
\]