Fluid Mechanics, SG2214, HT2013
September 30, 2013

Exercise 8: Exact solutions to energy equation

Example 1: Energy Dissipation in Poiseuille Flow

a) Calculate the dissipation function for the plane Poiseuille flow where

$$u = \frac{P}{2\,\mu}(h^2 - y^2), \quad v = w = 0,$$

or in terms of the bulk velocity U

$$u = \frac{3U}{2h^2}(h^2 - y^2), \quad v = w = 0.$$

The mass-flow rate through the channel is

$$Q = \int_{-h}^{h} u dy = 2Uh\ .$$

The dissipation function is defined as (dissipation to heat due to viscous stresses)

$$\Phi = \tau_{ij}\frac{\partial u_i}{\partial x_j}.$$

For incompressible flows, it can be re-written as

$$\Phi = \tau_{ij}\frac{\partial u_i}{\partial x_j} = 2\mu e_{ij}(e_{ij} + \xi_{ij}) = 2\mu e_{ij}e_{ij},$$

where we used the fact that $e_{ij}\xi_{ij} = 0$.

The deformation tensor for the Poiseuille flow becomes $e_{ij} = 1/2 \begin{bmatrix} 0 & \frac{\partial u}{\partial y} & 0 \\ \frac{\partial u}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and therefore

$$\Phi = 2\mu\left[\left(\frac{1}{2}\frac{\partial u}{\partial y}\right)^2 + \left(\frac{1}{2}\frac{\partial u}{\partial y}\right)^2\right] = \mu\left(\frac{\partial u}{\partial y}\right)^2.$$

b) Calculate the total dissipation for unit area

$$\phi = \int_{-h}^{h} \Phi\, dy = \int_{-h}^{h} \mu\left(-\frac{3U}{h^2}y\right)^2 dy = \frac{6\mu U^2}{h}.$$

c) Write the mechanical energy equation for this flow. Integrate over the channel width and relate the total dissipation ϕ to the pressure gradient and the mass flux.

The mechanical energy equation is obtained by multiplying the Navier-Stokes equations by u_i (the energy is $\rho(1/2)u_iu_i$). One gets

$$\rho\frac{D}{Dt}\left(\frac{1}{2}u_iu_i\right) = \rho F_iu_i - u_i\frac{\partial p}{\partial x_i} + u_i\frac{\partial \tau_{ij}}{\partial x_j}.$$

Considering the Poiseuille flow and re-writing the last term as

$$u_i\frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial u_i\tau_{ij}}{\partial x_j} - \Phi,$$

the energy equation reduces to

$$0 = uP + \frac{\partial}{\partial y}(u\tau_{xy}) - \Phi.$$

Integrating across the channel each term in the expression above, one obtains for the first term

$$\int_{-h}^{h} uP dy = P\int_{-h}^{h} u dy = QP,$$

where Q is the flow rate. This term represents the work rate by pressure forces.
The second term

$$\int_{-h}^{h} \frac{\partial}{\partial y}(u\tau_{xy})\, dy = [(u\tau_{xy})]_{-h}^{h} = 0$$

due to the no-slip boundary conditions.
The third term is the total dissipation $\phi = \int_{-h}^{h} \Phi\, dy$ defined above. Summarising

$$0 = QP - \int_{-h}^{h} \Phi\, dy.$$

One can check the results, using the expression for ϕ obtained in b). Just recall that

$$Q = \int_{-h}^{h} u dy = 2Uh,$$

and the pressure gradient can be expressed in terms of U as $P = \frac{3\mu U}{h^2}$. Therefore $QP = 6\mu U^2/h = \phi$.

Example 2: Exact solution for energy equation

Consider plane Poiseuille flow in a straight channel with walls at $y = \pm h$. The temperature at the lower wall is $T(-h) = T_W + \Delta T$, whereas the upper wall is at $T(h) = T_W$. The velocity field is

$$u = \frac{3U}{2}(h^2 - y^2), \quad v = w = 0.$$

a) Derive and plot the temperature distribution.

Let us consider the energy equation for incompressible fluid.

$$\rho_0 c_p \frac{D}{Dt}T = \nabla \cdot (K\,\nabla T) + \Phi$$

In this case, the equation of state is simply $\rho = \rho_0$, and $c_p = c_v$. We also assume K to be independent of the temperature T.
We can therefore assume a steady solution and a fully developed field: $\frac{\partial}{\partial t}T = 0$; $\frac{\partial}{\partial x}T = 0$. The material derivative term is then

$$\frac{D}{Dt}T = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = 0 + u \cdot 0 + 0 \cdot \frac{\partial T}{\partial y} = 0,$$

while the diffusion term reduces to $K\,\nabla^2 T = K\frac{\partial^2 T}{\partial y^2}$. Finally, as shown in recitation 5, the dissipation function Φ reduces for this case to

$$\Phi = 2\mu e_{ij}e_{ij} = 2\mu\left[\left(\frac{1}{2}\frac{\partial u}{\partial y}\right)^2 + \left(\frac{1}{2}\frac{\partial u}{\partial y}\right)^2\right] = \mu\left(\frac{\partial u}{\partial y}\right)^2.$$

Summarising the energy equations to be solved is

$$0 = \frac{\partial^2 T}{\partial y^2} + \frac{\mu}{K}\left(\frac{\partial u}{\partial y}\right)^2, \tag{1}$$

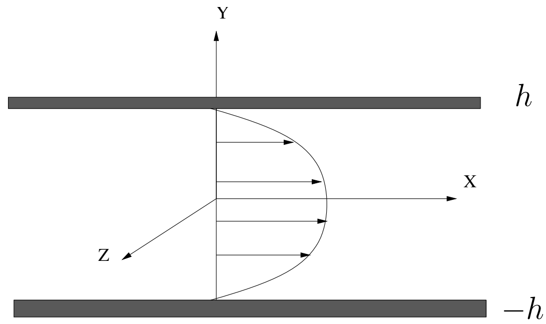

Figure 1: Coordinate system for plane Poiseuille flow.

with boundary conditions

$$\begin{cases} T(y=-h) &= T_W + \Delta T \\ T(y=h) &= T_W \end{cases}$$

Let us introduce dimensionless variables $y^* = \frac{y}{h}$, $T^* = \frac{T-T_W}{\Delta T}$ and $u^* = \frac{u}{U}$. Re-writing equation (1), one obtains

$$0 = \frac{\partial^2 T^*}{\partial y^{*2}} + \frac{\mu U^2}{K\Delta T}\left(\frac{\partial u^*}{\partial y^*}\right)^2. \quad (2)$$

The following adimensional numbers can then be introduced

$$\frac{\mu U^2}{K\Delta T} = \frac{\mu c_p}{K}\frac{U^2}{c_p\Delta T}.$$

The first term can be written as

$$\frac{\mu c_p}{K} = \frac{\nu}{K/(\rho c_p)} = \frac{\nu}{\kappa} = Pr,$$

where Pr denotes the Prandtl number, the ratio between the kinematic viscosity ν and the thermal diffusivity κ. $\frac{U^2}{c_p\Delta T} = E$ is the Eckert number defined mainly for fluids. Indeed, the same expression can be rewritten for gases in terms of the Mach number M and the adiabatic constant γ as

$$\frac{U^2}{c_p\Delta T} = \frac{M^2 a_w^2}{c_p\Delta T} = \frac{M^2\gamma R T_W}{c_p\Delta T} = (\gamma-1)\frac{M^2 T_W}{\Delta T}.$$

Summarising the energy equations in adimensional form is

$$0 = \frac{\partial^2 T^*}{\partial y^{*2}} + Pr\, E\left(\frac{\partial u^*}{\partial y^*}\right)^2, \quad (3)$$

with boundary conditions

$$\begin{cases} T(y^*=-1) &= 1 \\ T(y^*=1) &= 0 \end{cases}$$

Recalling the expression for the velocity field in adimensional variables $u^* = \frac{3}{2}\left(1-y^{*2}\right)$, $\frac{\partial u^*}{\partial y^*} = -3y^*$ and $\Phi = 9y^{*2}$. Therefore equation (3) can be integrated to yield

$$\frac{\partial^2 T^*}{\partial y^{*2}} + Pr\, E\, 9y^{*2} = 0,$$
$$\frac{\partial T^*}{\partial y^*} + Pr\, E\,\frac{9}{3}y^{*3} = C,$$
$$T^* + Pr\, E\,\frac{3}{4}y^{*4} = Cy^* + D.$$

C and D are determined imposing the boundary conditions at $y^* = \pm 1$.

$$D = \frac{1}{2}\left(1 + \frac{3}{2}PrE\right); \; C = -\frac{1}{2}.$$

Finally the temperature field can be written as

$$T^* = \frac{1}{2} + \frac{3}{4}PrE - \frac{1}{2}y^* - \frac{3}{4}PrE\left(y^{*4}\right)$$
$$T^* = \frac{1}{2}\left(1 - y^*\right) + \frac{3}{4}PrE\left(1 - y^{*4}\right),$$

and in dimensional form

$$T = 1 + \frac{1}{2}\frac{\Delta T}{T_W}\left(1 - \frac{y}{h}\right) + \frac{3}{4}Pr\frac{U^2}{c_p T_W}\left(1 - \left(\frac{y}{h}\right)^4\right).$$

The solution is composed of two parts, the first, linear in y, is the temperature distribution one would obtain in the presence of a temperature difference between the two walls. The second, fourth order contribution, is the heating due to dissipation in the fluid.

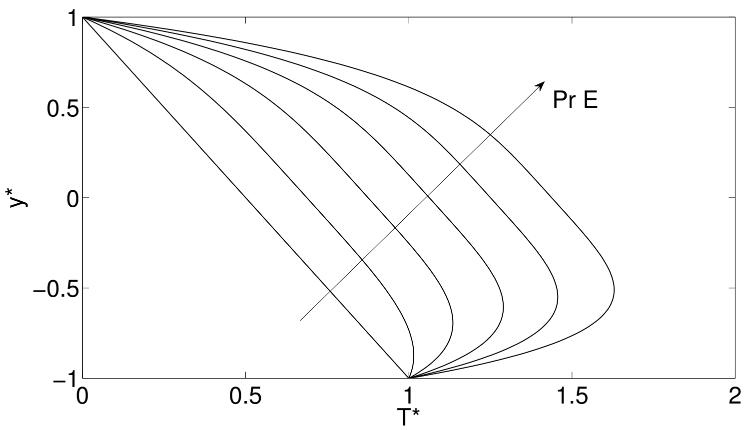

Figure 2: Temperature distribution for increasing PrE.

b) Write the heat flux at the wall. Determine the value of PrE for which the heat flux at the lower wall is zero.

$$\frac{\partial T^*}{\partial y^*}|_{y^*=-1} = \left[-\frac{1}{2} + \frac{3}{4}PrE\left(-4y^{*3}\right)\right]_{y^*=-1} = \frac{1}{2}(-1 + 6PrE).$$

The wall heat flux is

$$q_y = -K\frac{\partial T}{\partial y}|_{y=-h} = -\frac{K}{h}\Delta T\frac{\partial T^*}{\partial y^*}|_{y^*=-1} = -\frac{K}{h}\Delta T\frac{1}{2}(-1 + 6PrE).$$

Normalising

$$\frac{q_y}{\rho c_p U \Delta T} = -\frac{K\Delta T}{h\rho c_p U \Delta T}\frac{1}{2}(-1 + 6PrE) =$$
$$-\frac{K/(\rho c_p)}{hU}(-1/2 + 3PrE) = -\frac{\kappa}{Uh}(-1/2 + 3PrE) = -\frac{1}{Re}\frac{1}{Pr}(-1/2 + 3PrE).$$

Thus $q_y = 0$ if $PrE = 1/6$.