Short notes and exercises on the "*Incompressible Kelvin-Helmholtz instability*"

Jean-Christophe Loiseau [1], KTH 2015.

A mechanistic view of the Kelvin-Helmholtz instability

These two exercises aim at providing the student with a basic understanding of the physics of the Kelvin-Helmholtz instability. This instability, initially studied theoretically by Helmholtz (1868) and Lord Kelvin (1871), has proven to be a generic instability in a wide variety of shear flows at large Reynolds numbers. Based on physical arguments, two relatively simple explanations can be given to explain this basic instability mechanism. The following explanation, based on the vortex sheet modelisation of the problem (see exercise 1), is taken from Charru, *Hydrodynamic Instabilities*, Cambridge University Press.

> *The Kelvin-Helmholtz instability mechanism can be explained as a sort of "Bernoulli effect". Let us consider the flow in a reference frame moving at the average speed (figure 1)* [...] *and the speed of the perturbations vanishes at infinity. Above a perturbation $\eta > 0$ of the shear layer, the fluid is accelerated owing to the fact that the cross-sectional area perpendicular to the flow is decreased. This perturbation plays a role to a depth of order k^{-1}, the only length scale in the problem, on either side of the interface. The order of magnitude of the velocity perturbation is therefore such that $\eta\Delta U \simeq u/k$ owing to the incompressibility of the fluid. This velocity excess above a crest leads to a pressure decrease of order $p \simeq -\rho\Delta U u$ according to the linearized Bernoulli theorem, so that the pressure difference accross the interface amplifies the perturbation.*

Another explanation has been given by Batchelor (1967) based on vorticity considerations. Though it will not be reported here for the sake of conciseness, students are strongly encourage to careful read it in Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press 1967.

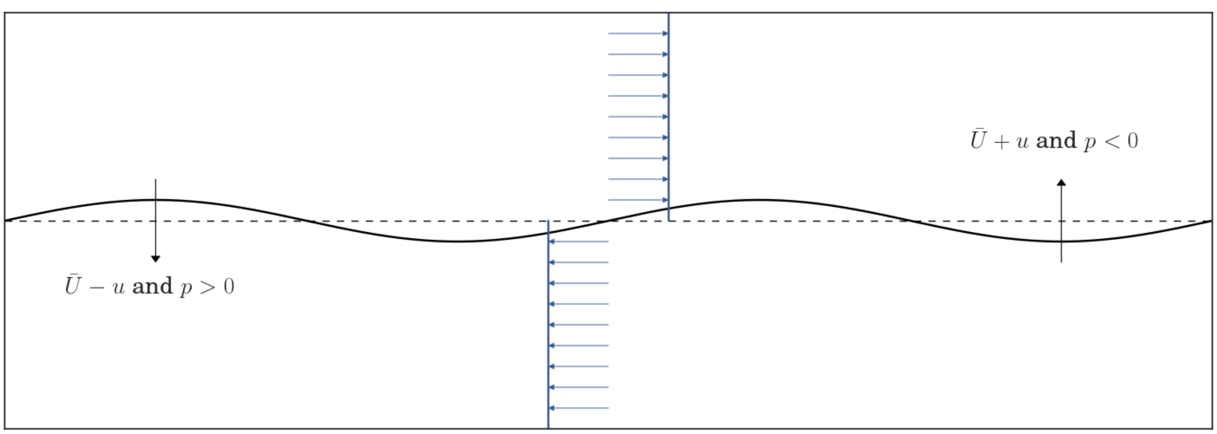

FIGURE 1 – Mechanistic representation of the Kelvin-Helmholtz observed in a reference frame moving at the average speed of the two streams. Adapated from Charru, *Hydrodynamic Instabilities*, Cambridge University Press.

1. Contact address : loiseau@mech.kth.se

Exercise 1 : Linear instability of a vortex sheet

The aim of this exercise is to give you some insights into the physics of the Kelvin-Helmholtz instability. This instability can usually be observed along the shear layer created by two joining streams of fluids. In the rest of this exercise, we furthermore assume that the two streams have the same constant density ρ. As a first approximation, ignoring the viscous effects compared to the intertial ones (*i.e.* $Re \to \infty$), the equations governing the dynamics of the flow are the incompressible Euler equations given by

$$\begin{aligned} &\nabla \cdot \mathbf{U} = 0 \\ &\partial_t \mathbf{U} = -(\mathbf{U} \cdot \nabla)\mathbf{U} - \nabla P \end{aligned} \tag{1}$$

The Euler equations admit solutions of the form $\mathbf{U}_b(\mathbf{x}, t) = (U_b(y), 0, 0)^T$ and $P_b(\mathbf{x}, t) = P_b$, with $U_b(y)$ being possibly a discontinuous function of y. Such solutions correspond to uni-directional flows being stationary and homogeneous both in the streamwise direction x and in the span-wise one z. Given such a base flow, the linearized Euler equations governing the dynamics of infinistesimally small perturbations $(\mathbf{u}, p)^T$ evolving onto $(\mathbf{U}_b, P_b)^T$ read

$$\begin{aligned} &(\partial_t + U_b \partial_x)u + U_b' v = -\partial_x p \\ &(\partial_t + U_b \partial_x)v = -\partial_y p \\ &(\partial_t + U_b \partial_x)w = -\partial_z p \\ &\partial_x u + \partial_y v + \partial_z w = 0 \end{aligned} \tag{2}$$

Since the aim of this exercise is to determine the instability condition for which the base flow $\mathbf{U}_b$ is linearly unstable, only two-dimensional perturbations given by $\mathbf{u} = (u, v, 0)^T$ can be considered thanks to the Squire theorem. Introducing the stream function ψ given by

$$u = \partial_y \psi \text{ and } v = -\partial_x \psi \tag{3}$$

and eliminating the pressure by cross-differentiation of the Euler euations, one obtains

$$(\partial_t + U_b \partial_x)\nabla^2 \psi - U_b'' \partial_x \psi = 0 \tag{4}$$

This equation is autonomous in time and in the streamwise direction x. As a consequence, its solutions can be sought in the form of normal modes

$$\begin{aligned} \psi(x, y, t) &= \frac{1}{2}\left(\hat{\psi}(y) \exp\left[i(kx - \omega t)\right] + c.c\right) \\ &= \Re\left(\hat{\psi}(y) \exp\left[ik(x - ct)\right]\right) \\ &= \left|\hat{\psi}(y)\right| \cos\left[k(x - c_r t)\right] \mathrm{e}^{kc_i t} \end{aligned} \tag{5}$$

where $c.c$ stands for complex conjugate, k is the streamwise wavenumber of the perturbation and $\omega = k(c_r + ic_i)$ its complex circular frequency. Introducing this normal mode ansatz into equation (4) yields to the *Rayleigh equation*

$$\left(U_b - \frac{\omega}{k}\right)\left(\partial_y^2 \hat{\psi} - k^2 \hat{\psi}\right) - U_b'' \hat{\psi} = 0 \tag{6}$$

associated to vanishing boundary conditions, *i.e.* $\hat{\psi}(y \to \pm\infty) = 0$.

As a first step toward our understanding of the Kelvin-Helmholtz instability, the velocity profile induced by a mixing layer will be approximated using a relatively crude model : the vortex sheet. The corresponding velocity profile is given by

$$U(y) = \begin{cases} U_1 \text{ for } y < 0 \\ U_2 \text{ for } y > 0 \end{cases} \tag{7}$$

Such a discontinuous velocity profile is depicted on figure 1.

Question 1 :

From the expression given for the vortex sheet velocity profile, one can easily see that $U_b'' = 0$. The Rayleigh equation (6) thus simplifies to

$$(U_b - c)\left(\partial_y^2 \hat{\psi} - k^2 \hat{\psi}\right) = 0 \tag{8}$$

In the rest of this exercise, we will assume that $(U_b - c) \neq 0$. Such assumption leads us to neglect the problem of *critical layers* which is beyond the scope of the present course. Equation (8) finally reduces to the simple second order differential equation

$$\partial_y^2 \hat{\psi} - k^2 \hat{\psi} = 0 \tag{9}$$

The reduced discriminant of such an equation is : $r^2 - k^2 r = 0$. Consequently, it appears obvious that equation (9) has solutions of the form

$$\hat{\psi}_j = A_j \mathrm{e}^{-ky} + B_j \mathrm{e}^{ky} \tag{10}$$

Question 2 :

We consider streamwise wavenumbers k such that $k \in \mathbb{R}^+$. It is easy to show that if y tends to $-\infty$ (respectively $+\infty$), then the general solution (10) diverges. The only possiblity to fulfill the vanishing boundary conditions is to impose $A_1 = 0$ (respectively $B_2 = 0$). We are then left with

$$\begin{aligned} \hat{\psi}_1 &= B_1 \mathrm{e}^{ky} \\ \hat{\psi}_2 &= A_2 \mathrm{e}^{-ky} \end{aligned} \tag{11}$$

Question 3 :

The two jump conditions are to be applied at the interface located at $y_0 = 0$. They express the continuity of the normal velocity to the interface and the normal force balance. For a complete derivation of these jump conditions, please look at F. Charru, *Hydrodynamic Instabilities*, page 115. These two jump conditions give

$$\begin{aligned} &\frac{A_2}{U_2 - c} - \frac{B_1}{U_1 - c} = 0 \\ &- k(U_2 - c)A_2 - (+k)(U_1 - c)B_1 = 0 \end{aligned} \tag{12}$$

After some simple manipulations, this linear system can be rewritten in an equivalent matrix form

$$\begin{bmatrix} (U_1 - c) & -(U_2 - c) \\ (U_2 - c) & (U_1 - c) \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \mathbf{0} \tag{13}$$

Question 4 :

The linear system (13) has a non-trivial solution provided that

$$\det\left(\begin{bmatrix} (U_1 - c) & -(U_2 - c) \\ (U_2 - c) & (U_1 - c) \end{bmatrix} \right) = 0 \tag{14}$$

After some manipulations, this condition can be re-written as a second degree polynomial equation

$$c^2 - (U_1 + U_2)c + \frac{U_1^2 + U_2^2}{2} = 0 \tag{15}$$

Introducing the following notations

$$\Delta U = U_1 - U_2 \text{ and } \bar{U} = \frac{U_1 + U_2}{2}$$

where ΔU characterizes the shear applied at the interface and $\bar{U}$ is the mean velocity, it is easy to show that equation (15) admits the following complex conjugate solutions

$$c_{\pm} = \frac{\omega}{k} = \bar{U} \pm i\frac{\Delta U}{2} \tag{16}$$

Regarding the temporal stability of the vortex sheet model toward perturbations having real wavenumbers, the speed c_r and temporal growth rate $\omega_i = \pm k c_i$ of these two modes are given by

$$c_r = \bar{U} \text{ and } \omega_i = \pm k\frac{\Delta U}{2}$$

Question 5 :

From the expressions derived in question 4, it is clear that the two instability waves have the same speed (they are not dispersive). Moreover, the temporal growth rate ω_i linearly depends on ΔU indicating that, provided $\Delta U \neq 0$, the vortex sheet model is linearly unstable for all wavenumber no matter how small the velocity difference between the two streams. It has to be noted also that, for a given vortex sheet profile (fixed ΔU), the temporal growth rate increases linearly with the streamwise wavenumber of the perturbation, as depicted on figure 3. This behaviour leads to an unphysical conclusion that the growth rate of the unstable perturbation is unbounded at large wavenumbers (small wavelengths) as a consequence of all the effects of viscous diffusion being neglected in the present model.

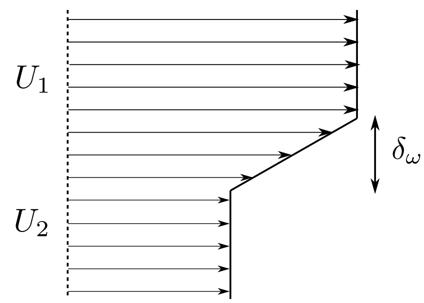

Figure 2 – Example of a piecewise linear mixing layer profile.

Exercise 2 : The piecewise-linear mixing layer profile

In order to overcome the limitations of the vortex sheet model, one has to include a length scale in the problem. Owing to diffusion by viscosity, a velocity discontinuity cannot be maintained and a more realistic model of base flow has to include a shear layer connecting the two uniform streams in a continuous manner. In the rest of this exercise, we will consider the following piecewise linear profile

$$U(y) = \begin{cases} U_1 & \text{for } y > \delta_\omega/2 \\ \bar{U} + \dfrac{\Delta U}{\delta_\omega} y & \text{for } |y| \leq \delta_\omega/2 \\ U_2 & \text{for } y < -\delta_\omega/2 \end{cases} \tag{17}$$

Such a velocity profile is depicted on figure 2. This simple model has been solved analytically by Lord Rayleigh in 1880.

Question 1 :

It has to be noted that, by using the piecewise linear velocity profile (17), we assume that the base flow does not evolve neither in time nor in the streamwise direction x. In reality however, due to viscous diffusion, the vorticity thickness of the shear layer will increase as time passes. We can thus define a characteristic viscous time-scale given by

$$\tau_\delta = \frac{\delta_\omega^2}{\nu} \tag{18}$$

Moreover, we have seen in the previous exercise that the Kelvin-Helmholtz instability is a shear-driven instability. As a consquence, one can define a characteristic time-scale for the instability as

$$\tau_{KH} = \frac{\delta_\omega}{\Delta U} \tag{19}$$

As a consequence, provided $\tau_{KH} << \tau_\omega$, the instability evolves on a time-scale much smaller than the diffusive time-scale of the shear layer. As a first approximation, it can be assumed that the instability sees the local vorticity thickness as being constant over time. We will moreover assume that the growth of this vorticity thickness in the streamwise direction is slow such that we can use the parallel flow assumption.

Thanks to all these different approximations, the governing equations for the perturbation are given by the linearized Euler equations. Moreover, thanks to the Squire theorem, only two-dimensional perturbations need to be considered when looking for the linear instability condition of the flow. As a consequence, the linear stability analysis of the piecewise-linear velocity profile (17) can be investigated using the two-dimensional Rayleigh equation.

Question 2 :

As for the previous exercise, it is easy to show that, given the boundary conditions, the Rayleigh equation has solutions of the form

$$\begin{cases} \hat{\psi}_1 = A_1 \mathrm{e}^{-ky} \\ \hat{\psi}_\delta = A_\delta \mathrm{e}^{-ky} + B_\delta \mathrm{e}^{ky} \\ \hat{\psi}_2 = B_2 \mathrm{e}^{ky} \end{cases} \tag{20}$$

Question 3 :

Expliciting the first jump condition (continuity of velocity) at $y_0 = \delta_\omega/2$ and $y_0 = -\delta_\omega/2$ gives

$$\begin{cases} A_1 = A_\delta + B_\delta \mathrm{e}^{k\delta_\omega} \\ B_2 = A_\delta \mathrm{e}^{k\delta_\omega} + B_\delta \end{cases} \tag{21}$$

Using these relations when expliciting the second jump condition (balance of normal forces) gives the following linear system

$$\begin{cases} \dfrac{\Delta U}{\delta_\omega} \mathrm{e}^{-k\delta_\omega/2} A_\delta + \left(\dfrac{\Delta U}{\delta_\omega} - 2k(U_1 - c) \right) \mathrm{e}^{k\delta_\omega/2} B_\delta = 0 \\ \left(-2k(U_2 - c) - \dfrac{\Delta U}{\delta_\omega} \right) \mathrm{e}^{k\delta_\omega/2} A_\delta - \dfrac{\Delta U}{\delta_\omega} \mathrm{e}^{-k\delta_\omega/2} B_\delta = 0 \end{cases} \tag{22}$$

Finally, this system of linear equations can be re-written in an equivalent matrix form as follows

$$\begin{bmatrix} -\dfrac{\Delta U}{\delta_\omega} \mathrm{e}^{-k\delta_\omega/2} & \left(2k(U_1 - c) - \dfrac{\Delta U}{\delta_\omega} \right) \mathrm{e}^{k\delta_\omega/2} \\ \left(2k(U_2 - c) - \dfrac{\Delta U}{\delta_\omega} \right) \mathrm{e}^{k\delta_\omega/2} & \dfrac{\Delta U}{\delta_\omega} \mathrm{e}^{-k\delta_\omega/2} \end{bmatrix} \begin{bmatrix} A_\delta \\ B_\delta \end{bmatrix} = \mathbf{0} \tag{23}$$

Question 4 :

The linear system (23) has non-trivial solutions if its determinant is null, that is

$$\frac{\Delta U^2}{\delta_\omega^2} \mathrm{e}^{-k\delta_\omega} + \left(2k(U_1 - c) - \frac{\Delta U}{\delta_\omega} \right) \left(2k(U_2 - c) + \frac{\Delta U}{\delta_\omega} \right) \mathrm{e}^{k\delta_\omega} = 0 \tag{24}$$

After some manipulations, this equation finally reads

$$\Delta U^2 \left(\mathrm{e}^{-2k\delta_\omega} - (k\delta_\omega - 1)^2 \right) + 4(k\delta_\omega)^2 (\bar{U} - c)^2 = 0 \tag{25}$$

Figure 3 depicts the evolution of the speed c_r and of the temporal growth rate kc_i of the perturbation with respect to its streamwise wavenumber. For $(k\delta_\omega - 1)^2 > \mathrm{e}^{-2k\delta_\omega}$ (*i.e.* $k\delta_\omega > k_c\delta_\omega$ where

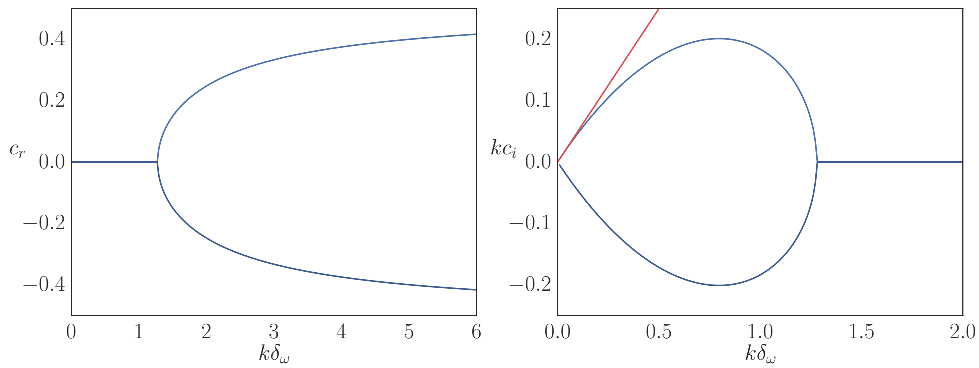

FIGURE 3 – (a) Speed and (b) temporal growth rate of the stable and unstable modes of the broken-line mixing layer velocity profile. The red line on (b) depicts the dispersion relation of the vortex sheet studied in exercise 1.

k_c is a cut-off wavelength), the dispersion relation has two real roots $c_\pm$ and the perturbation is neither exponentially growing nor decaying, it is neutral. On the other hand, for $k\delta_\omega < k_c\delta_\omega$, the dispersion relation has two complex conjugate roots : one of the eigenmode is therefore exponentially growing while the other decays. As a consequence, the broken-line velocity profile considered is linearly unstable. It has to be noted finally that for $k\delta_\omega << 1$, *i.e.* perturbations of wavelength much larger than the vorticity thickness, we recover the trend given by the vortex sheet modelisation as could have been expected.