Conservation of Energy in Flatland

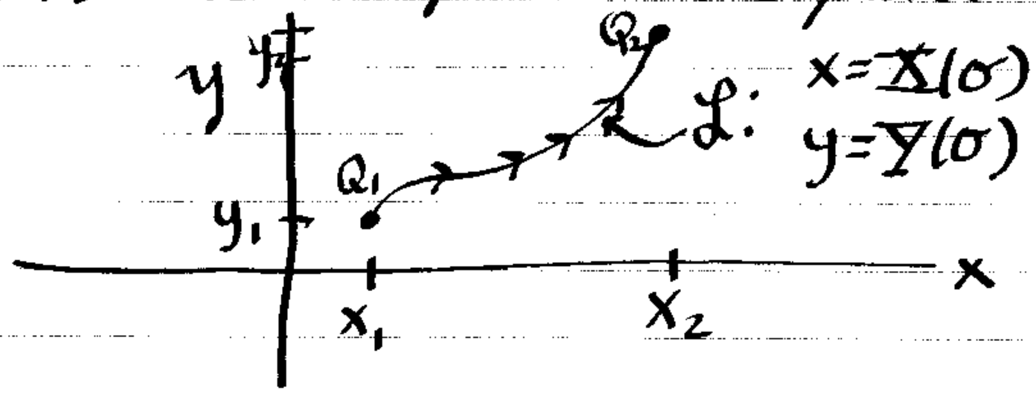
Basic applied forces such as gravity and even many more empirical forces such as those of a linear spring have a very special property – they are 'conservative'. To see what this means we first define the <u>work done</u> ~~by~~ a particle as the integrated force acting to move a particle over a distance.

$$\text{work} = \text{Real}\left(\int_{\mathcal{L}} F_a \, dZ^*\right)$$

Thus if $F_a = f_1 + i f_2$, $z = x + iy$.

$$\text{Work}(\mathcal{L}) = \int_{\mathcal{L}} (f_1 \, dx + f_2 \, dy).$$

where $\mathcal{L}$ is a path in the plane



Thus given a parametrized curve in flatspace

$$x = X(\sigma), \quad y = Y(\sigma)$$

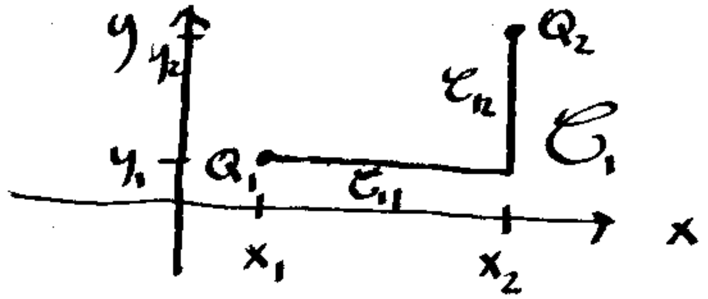
with $x_j = X(\sigma_j)$, $y_j = Y(\sigma_j)$ $\Big\}\, Q_j$

defining the curve $\mathscr{L}$ and its endpoints.

The work done is computed as

$$\text{work}(\mathscr{L}) = \int_{\sigma_1}^{\sigma_2} \left(f_1[X(\sigma), Y(\sigma)] \frac{dX}{d\sigma} + f_2[X(\sigma), Y(\sigma)] \frac{dY}{d\sigma} \right) d\sigma$$

As an example consider the work done by gravity in a uniform gravitational field on the path $\mathscr{C}_1$



$g \downarrow$

$f_1 = 0$

$f_2 = -mg$

$$\text{Work}(\mathscr{C}_1) = \text{Real}\left(\int_{\mathscr{C}} F_a \, dz^* \right)$$

$$= \text{Real}\left(\int_{\mathscr{C}_1} F_a \, dz^* + \int_{\mathscr{C}_2} F_a \, dz^* \right)$$

on $\mathscr{C}_1$, $dz^* = dx$, $y = y_1$, $x: x_1 \to x_2$

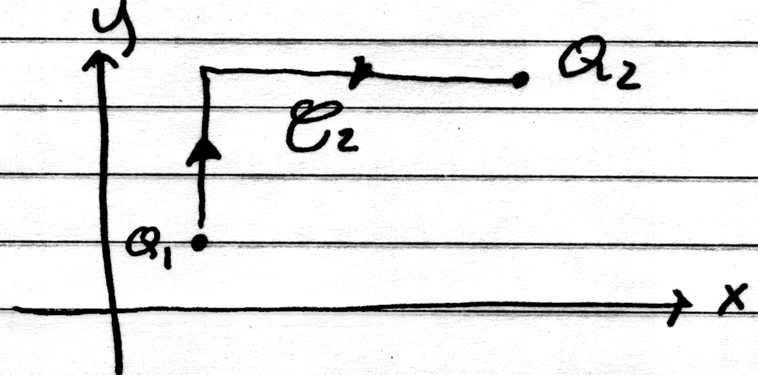
on $\mathscr{C}_2$, $dz^* = -i\,dy$, $x = x_2$, $y: y_1 \to y_2$

$$\text{Work} = \text{Real} \int_{y_1}^{y_2} (-i\,dy)(-i\,mg) =$$

$$-mg(y_2 - y_1)$$

If $y_2 > y_1$, $\text{Work}(\mathcal{C}_1) < 0$

which implies that the work done is on the particle, i.e. we must counter the gravitational force to raise the particle to an increased height!

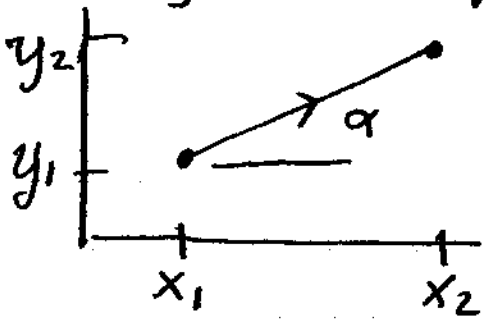


The reader can verify that the same result is obtained for the path $\mathcal{C}_2$! That is in this case we have

$$\text{Work}(\mathcal{C}_1) = \text{Work}(\mathcal{C}_2)$$

The remarkable fact is that we require the same work for two different paths!

Now lets try a third path.



On this path $z = x_1 + iy_1 + \sigma e^{i\alpha}$

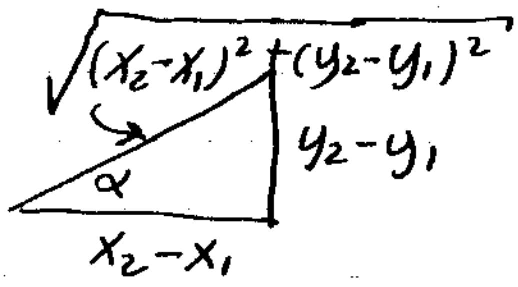
when $\sigma = 0$, $z = x_1 + iy_1 = z_1$

when $\sigma = \sigma_2$ $z = x_1 + i y_1 + \sigma_1 \cos\alpha + i \sigma_1 \sin\alpha$
$= z_2$

$\therefore$ $x_2 = x_1 + \sigma_2 \cos\alpha$
$y_2 = y_1 + \sigma_2 \sin\alpha$

so $\tan\alpha = \dfrac{y_2 - y_1}{x_2 - x_1}$

$\sigma_2 = \dfrac{y_2 - y_1}{\sin\alpha}$,



$$\sin\alpha = \frac{y_2 - y_1}{\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}} = \frac{\Delta y}{|\Delta z|}$$

$$\cos\alpha = \frac{x_2 - x_1}{\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}} = \frac{\Delta x}{|\Delta z|}$$

$$\therefore \quad z = z_1 + \sigma e^{i\alpha}$$

$$z_2 = z_1 + \sigma_2 e^{i\alpha}$$

$$dz = d\sigma\, e^{i\alpha}$$

$$\text{Work}(\mathcal{C}_3) = \text{Real} \int_0^{\sigma_2} (-i mg)\, e^{-\alpha i}\, d\sigma$$

$$= \text{Real}\left(-i mg\, \sigma_2\, e^{-i\alpha}\right)$$

$$= \text{Real}\left(-i mg\, \sigma_2 \cos\alpha - i mg\, \sigma_2 (-i \sin\alpha)\right)$$

$$= -mg\, \sigma_2 \sin\alpha$$

$$= -mg\, \frac{y_2 - y_1}{\sin\alpha} \sin\alpha = -mg\,(y_2 - y_1)$$

Hence for this case we have

$$\text{Work}(\mathcal{C}_1) = \text{Work}(\mathcal{C}_2) = \text{Work}(\mathcal{C}_3)$$

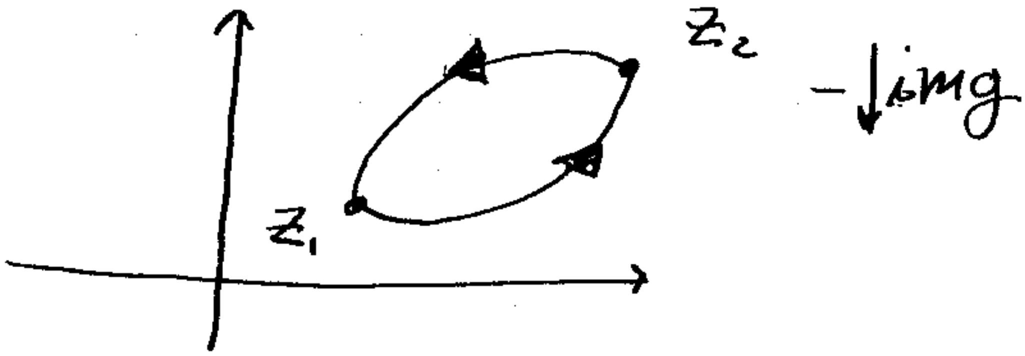
In fact, choosing <u>any</u> path between z_1 and z_2 will give the exact same result. For the chosen force $F = -\hat{j}mg$ the work integral is independent of the path. This allows us to conclude that for this force the work is only a function of the chosen end points and we can write that for any path between z_1 and z_2

$$\text{Work} = W(z_1, z_2)$$

As going from $z_2 \to z_1$ reverses the sign we have

$$W(z_2, z_1) = -W(z_1, z_2)$$

or $$W(z_1, z_1) = W(z_1, z_2) + W(z_2, z_1)$$
$$= 0.$$



Clearly the force $-img$ is special. A force that is independent of path is called <u>Conservative</u>.

We would like to know what condition such a force must obey to have this property!

~~Suppose the force~~

Define the (<u>Potential Energy</u>) $V(x,y)$

If the integral $W(\mathcal{C})$ is independant of path then

$$V(x,y) = -\,\mathrm{Real}\int_{z_{ref}}^{z} F_a \, dz^*$$

Now if $V(x,y)$ exists

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy$$

$$= -\,\mathrm{Real}\,(f_1 + i f_2)(dx - i\,dy)$$

$$= -f_1\,dx - f_2\,dy$$

which in turn implies that

$$f_1 = -\frac{\partial V}{\partial x} \quad , \quad f_2 = -\frac{\partial V}{\partial y}$$

$$\boxed{F_a = -\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) V}$$

Sometimes we write $\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \nabla$.

Now

$$\frac{\partial f_1}{\partial y} = -\frac{\partial^2 V}{\partial y \partial x} = -\frac{\partial^2 V}{\partial x \partial y} = -\frac{\partial}{\partial x}\frac{\partial V}{\partial y}$$

$$= -\frac{\partial}{\partial x}(-f_2) = \frac{\partial f_2}{\partial x}$$

$$\Rightarrow \boxed{\mathrm{Imag.}(\nabla^* F) = 0}$$

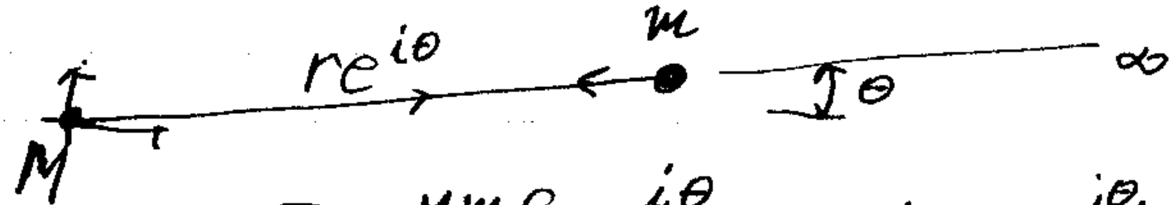
Which is thus a necessary condition for the existance of $V(x,y)$.

<u>summary</u> $W(C) = \mathrm{Real}\int_C F\,dz^*$
if W indep. of path $-V(x,y) = W(x,y)$

and $V = \mathrm{Real}\int_{z_{\mathrm{Reference}}}^{z} F\,dz^*$, $F = -\nabla^* V$

and $\mathrm{Imag.}\,\nabla^* F = 0$

Example: Newtonian Gravity



$$F = -\frac{MmG}{r^2} e^{i\theta} \quad , \quad dz = \cancel{r}\, e^{i\theta} dr$$

We leave it for later to show path independance.

$$-\int_{r=\infty}^{r} F\, dz^* = -\int_{\infty}^{r} -\frac{MmG}{r^2} e^{i\theta} e^{-i\theta} dr$$

$$= -\int_{\infty}^{r} -\frac{MmG}{r^2} e^{i\theta - i\theta} dr = -\frac{MmG}{r}\Big]_{\infty}^{r}$$

$$\therefore \quad V = -\frac{MmG}{r}$$

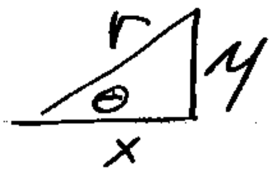
in polar coord.

$$\nabla = e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right)$$

$$\therefore \; -\nabla V = -e^{i\theta}(-1)\frac{\partial}{\partial r}\frac{MmG}{r}$$

$$\boxed{F = -\frac{MmG}{r^2} e^{i\theta}}$$

note $f_1 = -\frac{MmG}{r^2}\cos\theta \; , \; f_2 = -\frac{MmG}{r^2}\sin\theta$



$$f_1 = -\frac{MmG}{r^2}\frac{x}{r} \quad , \quad f_2 = -\frac{MmG}{r^2}\frac{y}{r}$$

$$\frac{1}{MmG}\frac{\partial f_1}{\partial y} = -\frac{x}{r^2}\frac{\partial r}{\partial y}$$

$$\frac{1}{MmG}\frac{\partial f_2}{\partial x} = -\frac{y}{r^2}\frac{\partial r}{\partial x}$$

$$r^2 = x^2 + y^2 \rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\therefore \quad \frac{1}{MmG}\frac{\partial f_1}{\partial y} = -\frac{xy}{r^3}, \quad \frac{1}{MmG}\frac{\partial f_2}{\partial x} = -\frac{yx}{r^3} \rightarrow \frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}$$

So the necessary condition is satisfied

Hence
Newtonian Gravity is Conservative
with

$$V = -GM\frac{m}{r}$$

$$F = -\frac{GMm}{r^2}e^{i\theta}$$

Now if

$$m\frac{d^2 z}{dt^2} = F_a = -\nabla V$$

$$m\ddot{z} + \nabla V = 0$$

$$\therefore \text{Real}\left(m\dot{z}^*\ddot{z} + \dot{z}^*\nabla V\right) = 0$$

if $x = x(t), y = y(t)$

$$\frac{d}{dt}V(x,t) = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}$$

$$= \text{Real}\,\dot{z}^*(\nabla V)$$

$$\text{Real}\left(\dot{z}^*\ddot{z}\right) = \dot{x}\ddot{x} + \dot{y}\ddot{y} = \frac{1}{2}\frac{d}{dt}\dot{x}^2 + \frac{1}{2}\frac{d}{dt}\dot{y}^2$$

$$\therefore \quad \frac{d}{dt}\left(\frac{m}{2}|\dot{z}|^2 + V\right) = 0$$

$$\boxed{\frac{1}{2}m|\dot{z}|^2 + V = \text{Constant} = E_0}$$

E_0 = Energy

For a Conservative Force Energy is conserved.