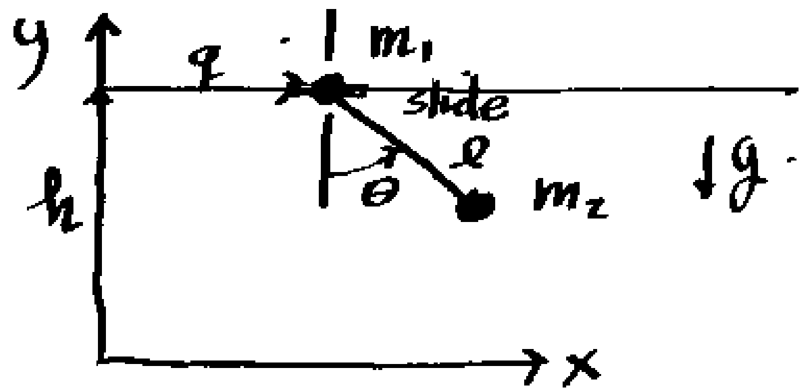
Examples of Lagranges Equations.

(A)



$$r_1 = ih + q \; ; \quad r_2 = r_1 - i\ell e^{i\theta}$$

$$\dot{r}_1 = \dot{q} \, , \quad \dot{r}_2 = \dot{q} - i^2 \ell\dot{\theta} e^{i\theta} = \dot{q} + \ell\dot{\theta} e^{i\theta}$$

$$T = \tfrac{1}{2} m_1 \dot{r}_1^{\,2} + \tfrac{1}{2} m_2 \dot{r}_2^{\,2}$$

$$\dot{r}_1^{\,2} = \dot{r}_1 \dot{r}_1^{*} = \dot{q}^2$$

$$\dot{r}_2^{\,2} = (\dot{q} + \ell\dot{\theta} e^{i\theta})(\dot{q} + \ell\dot{\theta} e^{-i\theta})$$

$$= \dot{q}^2 + (\dot{q}\ell\dot{\theta})(e^{i\theta} + e^{-i\theta}) + \ell^2 \dot{\theta}^2$$

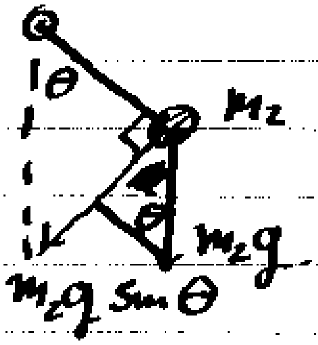
now use $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$

$$\dot{r}_2^{\,2} = \dot{q}^2 + 2\ell\dot{q}\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2$$

The only potential force is on the second mass m_2 and from previous work.

$$V_2 = -m_2 g \cos\theta$$

check $-\frac{\partial V_2}{\partial \theta} = -m_2 g \sin\theta$



hence

$$\boxed{L = \frac{1}{2}(m_1 + m_2)\dot{q}^2 + \frac{1}{2} m_2 \left(l^2\dot{\theta}^2 + 2l\dot{\theta}\dot{q}\cos\theta\right) + m_2 g \cos\theta}$$

$$\frac{\partial L}{\partial \dot{q}} = (m_1 + m_2)\dot{q} + m_2 l \dot{\theta}\cos\theta$$

$$\frac{\partial L}{\partial q} = 0$$

$$\therefore \boxed{\frac{d}{dt}\left((m_1 + m_2)\dot{q} + m_2 l \dot{\theta}\cos\theta\right) = 0}$$

$$\frac{\partial L}{\partial \dot\theta} = m_2\left(l^2\dot\theta + l\dot q\cos\theta\right)$$

$$\frac{\partial L}{\partial \theta} = -m_2 l\dot\theta\dot q\sin\theta - m_2 g\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = m_2 l^2\ddot\theta + m_2 l\ddot q\cos\theta - m_2 l\dot q\dot\theta\sin\theta$$

Therefore the equations of motion are,

from $\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q}$

$$\boxed{\frac{d}{dt}\left[(m_1+m_2)\dot q + m_2 l\dot\theta\cos\theta\right] = 0}$$

and from $\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial q}$

$$\boxed{\ddot\theta + \frac{\ddot q}{l}\cos\theta + \frac{g}{l}\sin\theta = 0}$$

Introduce the following notational changes

$$Q = q/l$$

$$M = (m_1+m_2)/m_2 = \frac{m_1}{m_2} + 1$$

$$\omega^2 = g/l$$

So the equations have the form

$$\begin{cases} \frac{d}{dt}\left(M\dot{Q} + \dot{\theta}\cos\theta \right) = 0 \\ \\ \ddot{\theta} + \ddot{Q}\cos\theta + \omega^2 \sin\theta = 0 \end{cases}$$

The first equation can be integrated at once to give

$$M\dot{Q} + \dot{\theta}\cos\theta = \text{Const.}$$

$$M\dot{Q} + \dot{\theta}\cos\theta = M\dot{Q}_0 + \dot{\theta}_0 \cos\theta_0$$

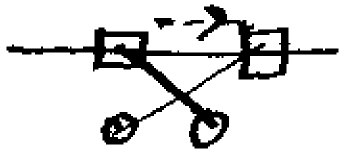
$$\frac{d}{dt}\left(MQ + \sin\theta\right) = M\dot{Q}_0 + \dot{\theta}_0 \cos\theta_0$$

$$\therefore \quad MQ + \sin\theta = \left(M\dot{Q}_0 + \dot{\theta}_0 \cos\theta_0\right) t + MQ_0 + \sin\theta_0$$

$$MQ + \sin\theta = \left[\frac{d(MQ + \sin\theta)}{dt}\right]_{t=0} t + \left(MQ + \sin\theta\right)_{t=0}$$

$\cancel{\text{if } Q_0 = \sin\theta_0 \; M\dot{Q}_0 + \dot{\theta}_0 \cos\theta_0}$

take $\dot{\theta}_0 = \dot{Q}_0 = 0$, $Q_0 = 0$



$$\boxed{MQ + \sin\theta = \sin\theta_0}$$

see saw motion

also $M\ddot{Q} + \ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta = 0$

or $\ddot{Q} = +\frac{1}{M}\left(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta\right)$

substitution then gives

$$\frac{1}{M}\left(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta\right)\cos\theta + \ddot{\theta} + \omega^2\sin\theta = 0$$

$$\left(1 - \frac{1}{M}\cos^2\theta\right)\ddot{\theta} + \frac{\cos\theta\sin\theta}{M}\dot{\theta}^2 + \omega^2\sin\theta = 0$$

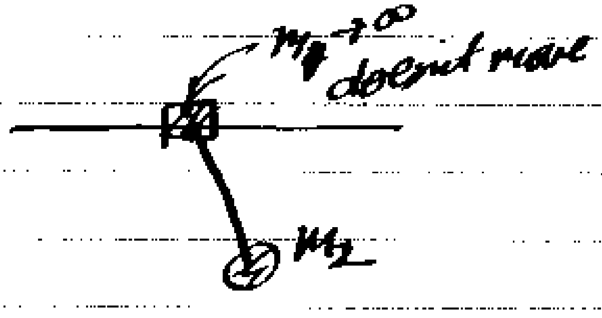
$$\boxed{\ddot{\theta} + \frac{\cos\theta\sin\theta}{M - \cos^2\theta}\dot{\theta}^2 + \frac{\omega^2 M\sin\theta}{M - \cos^2\theta} = 0}$$

$$M = \frac{m_1 + m_2}{m_2} = 1 + \frac{m_1}{m_2}$$

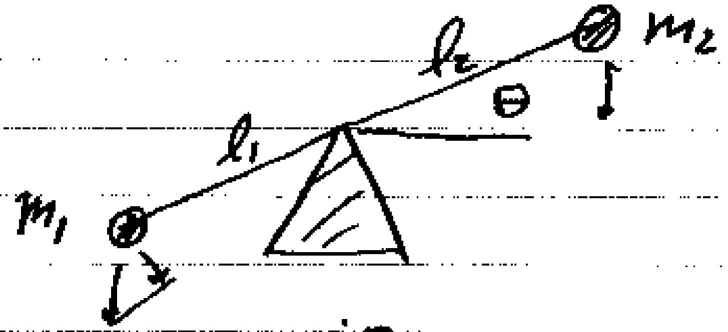
if $m_1 \to \infty$, $M \to \infty$

$\cancel{\ddot{\theta} + \cos\theta\sin\theta}$

$\ddot{\theta} + \omega^2\sin\theta = 0$



Example 13 (seesaw)



$\downarrow mg$

$$r_2 = l_2 e^{i\theta}, \quad r_1 = -l_1 e^{i\theta}$$

$$\dot{r}_2 = i l_2 \dot{\theta} e^{i\theta}, \quad \dot{r}_1 = -i l_1 \dot{\theta} e^{i\theta}$$

$$V_1 = l_1 m_1 g \sin\theta, \quad V_2 = -l_2 m_2 g \sin\theta$$

$$L = T - V$$

$$T = \frac{m_2}{2} l_2^2 \dot{\theta}^2 + \frac{m_1}{2} l_1^2 \dot{\theta}^2$$

$$V = (-l_1 m_1 g + l_2 m_2 g) \sin\theta$$

$$T - V = \frac{1}{2}\dot{\theta}^2 (m_1 l_1^2 + m_2 l_2^2) - (l_2 m_2 - l_1 m_1) g \sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = (m_1 l_1^2 + m_2 l_2^2)\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = (l_2 m_2 - l_1 m_1) g \cos\theta$$

$$\therefore \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = (m_1 l_1^2 + m_2 l_2^2)\ddot{\theta} + (l_1 m_1 - l_2 m_2) g \cos\theta$$

$$= 0$$

$$\cancel{\#} \quad \ddot{\theta} - \left(\frac{l_1 m_1 - l_2 m_2}{l_1^2 m_1 + l_2^2 m_2} \right) g \cos\theta = 0$$

$$\ddot{\theta} = \frac{m_1 l_1 - m_2 l_2}{l_1^2 m_1 + l_2^2 m_2} g \cos\theta$$

$m_1 l_1 > m_2 l_2 \qquad \dot{\theta} \uparrow$

$m_1 l_1 < m_2 l_2 \qquad \dot{\theta} \downarrow$

equalibrium when $m_1 l_1 = m_2 l_2$

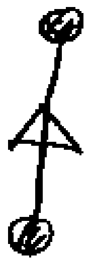
let $m_2 = M m_1$

$l_2 = L l_1$

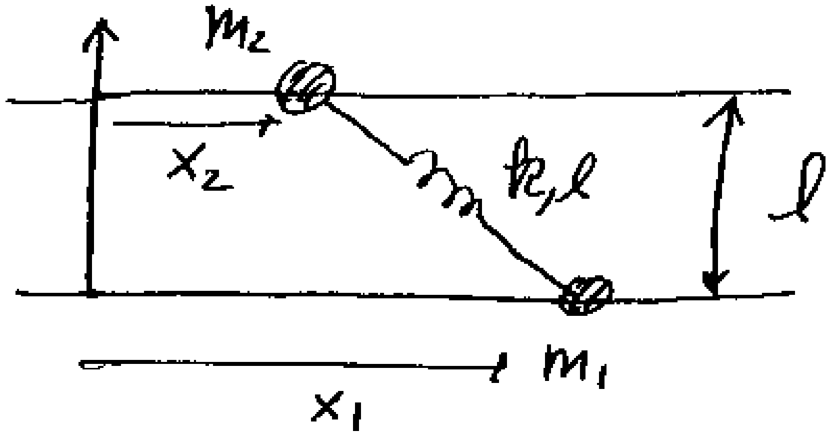
$$\ddot{\theta} = \frac{(1 - ML) m_1 l_1}{(1 + ML^2) m_1 l_1^2} g \cos\theta = 0$$

$$\ddot{\theta} = \left(\frac{1 - ML}{1 + ML^2} \right) \left(\frac{g}{l_1} \right) \cos\theta$$

another equalibrium when $\theta = \frac{\pi}{2}, -\frac{\pi}{2}, \cos\theta + 0$



Example C



$$\text{stretch} = \sqrt{\ell^2+(x_1-x_2)^2} - \ell$$

$$V = \frac{1}{2} k \left(\sqrt{\ell^2+(x_1-x_2)^2} - \ell \right)^2$$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$L = T - V = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k \left(\sqrt{\quad} - \ell \right)^2$$

$$\frac{\partial V}{\partial x_j} = k \left(\sqrt{\ell^2+(x_1-x_2)^2} - \ell \right) \left(\ell^2+(x_1-x_2)^2 \right)^{-1/2} \cancel{2}(x_1-x_2)(-1)^{j+1}$$

$$\therefore \quad \frac{\partial L}{\partial x_j} = (-1)^j k \frac{\left(\sqrt{\ell^2+(x_1-x_2)^2} - \ell \right)}{\sqrt{\ell^2+(x_1-x_2)^2}} (x_1 - x_2)$$

$$\frac{\partial L}{\partial \dot{x}_j} = m_j \ddot{x}_j$$

$$m_j \ddot{x}_j + (-1)^j \left\{ 1 - \frac{\ell}{\sqrt{(\ell^2 + (x_1 - x_2)^2)}} \right\} k (x_1 - x_2) = 0$$

we now consider the case where $|x_1 - x_2| << \ell$

<u>let</u> $\frac{x_1 - x_2}{\ell} = \Delta$

The second term in the above is thus

$$1 - (1 + \Delta^2)^{-1/2}$$

and by the binomial theorem this is

$$1 - \left\{ 1 - \tfrac{1}{2}\Delta^2 + \cdots \right\} = \tfrac{1}{2}\Delta^2 + \cdots$$

Therefore the equations of motion are approximatively: $j = 1, 2$

$$m_j \ddot{x}_j + \frac{(-1)^j}{2} k \frac{(x_1 - x_2)^3}{\ell^2} = 0$$

or if $k/m_j = \omega_j^2$:

$$\boxed{\ddot{x}_j + (-1)^j \frac{1}{2}\omega_j^2 \frac{(x_1 - x_2)^3}{\ell^2} = 0}$$

What is interesting here is that the equations are intrinsically non-linear.

$$\ddot{X}_1 + \frac{1}{2}\frac{\omega_1^2}{\ell^2}(X_2 - X_1)^3 = 0$$

$$\ddot{X}_2 + \frac{1}{2}\frac{\omega_2^2}{\ell^2}(X_1 - X_2)^3 = 0$$

We can choose units where
$\omega_1 = \sqrt{2}$, $\ell = 1$, $\omega_2 = \omega\sqrt{2}$

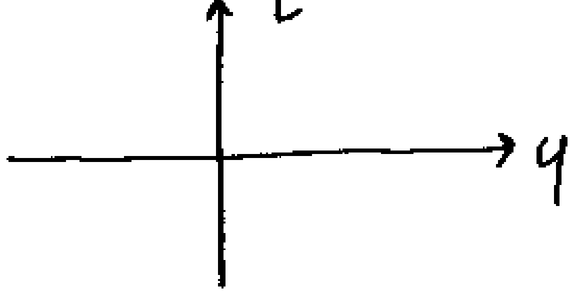
$$\ddot{X}_1 + (X_2 - X_1)^3 = 0$$
$$\ddot{X}_2 + \omega^2 (X_2 - X_1)^3 = 0$$

$$\frac{d^2}{dt^2}(X_2 - X_1) - (\omega^2 + 1)(X_2 - X_1)^3 = 0$$

if $y = X_2 - X_1$, $\omega^2 + 1 = K$

$$\boxed{\ddot{y} - Ky^3 = 0}$$

$$\dot{y} = \eta$$
$$\dot{\eta} = Ky^3$$



$$\frac{d\eta}{dy} = \kappa \frac{y^3}{\eta}$$

$$\eta\, d\eta - \kappa y^3 dy = 0$$

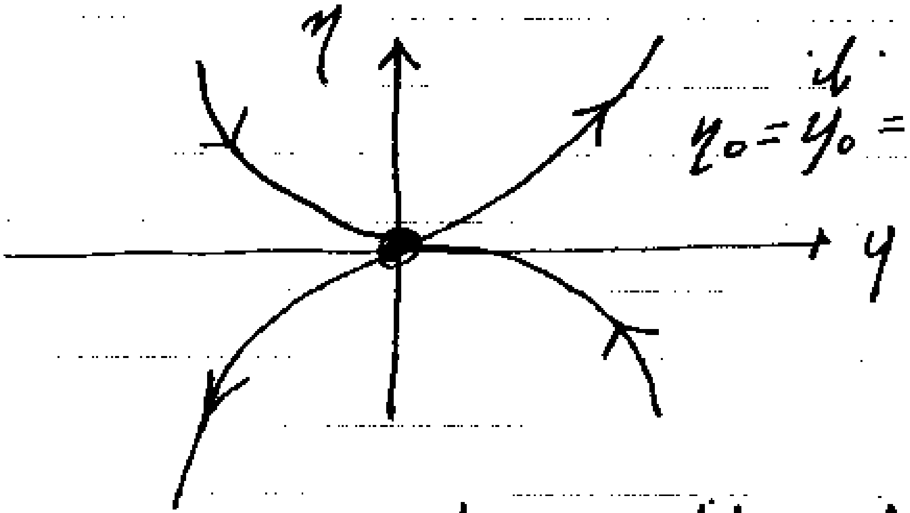
$$d\left(\frac{1}{2}\eta^2 - \frac{\kappa y^4}{4}\right) = 0$$

$\therefore \quad \eta^2 - \frac{\kappa}{2} y^4 = \text{Const.} \qquad$ let $\frac{\kappa}{2} = \lambda^2$

$$\eta^2 - \lambda^2 y^4 = \eta_0^2 - \lambda^2 y_0^4$$

$$(\eta - \lambda y^2)(\eta + \lambda y^2) = (\eta_0 - \lambda y_0^2)(\eta_0 + \lambda y_0^2)$$

two branches $\quad \eta = \lambda y^2, \quad \eta = -\lambda y^2$



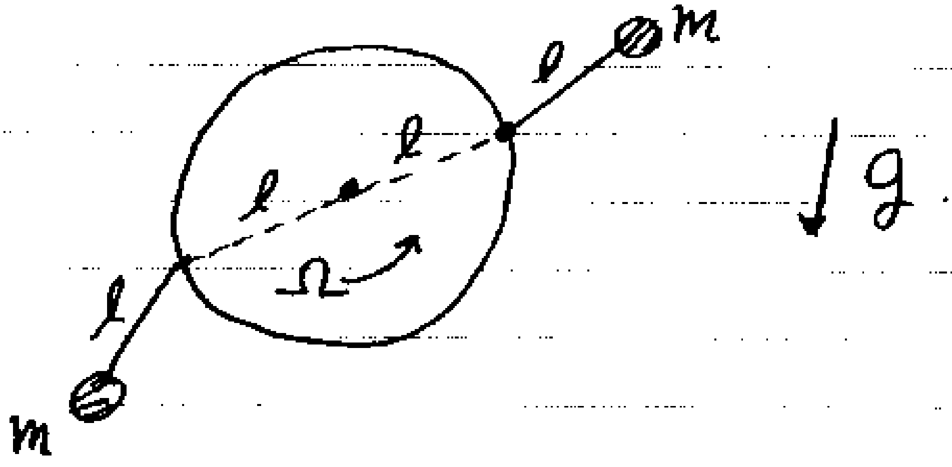
if

$\eta_0 = y_0 = 0$ or $\eta_0^2 = \lambda^2 y_0^4$

$\eta_0 = \lambda y_0^2$

$\eta_0 = -\lambda y_0^2$

shows stability depends on initial velocity



Choose units in which $l = m = g = 1$

$[l] = L$, $[m] = M$, $[g] = L/T^2$

$\Rightarrow T \to \sqrt{\frac{l}{g}}$ hence a result that should have units of velocity in SI units must be multiplied by

$$\frac{L}{T} = \frac{l\sqrt{g}}{\sqrt{l}} = \sqrt{gl}$$

if the result is an acceleration use

$$\frac{L}{T^2} = \frac{l\,g}{l} = g.$$

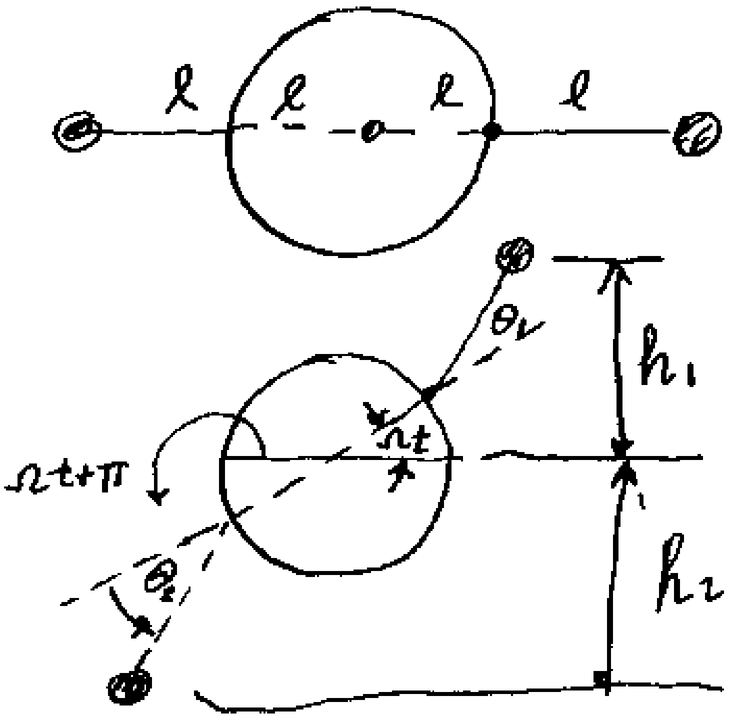
and for a force use

$$\frac{ML}{T^2} = mg. \quad \text{etc.}$$

$$r_1 = \ell e^{i\Omega t} + \ell e^{i(\Omega t + \theta_1)}$$

$$r_2 = \ell e^{i\Omega t} e^{i\pi} + \ell e^{i(\Omega t + \pi + \theta_2)}$$

using the reference configuration



$\theta_1 = \theta_2 = 0, \ t = 0$

$r_1 = 2\ell$

$r_2 = -2\ell$

$$r_1 = \ell e^{i\Omega t}(1 + e^{i\theta_1})$$

$$r_2 = -\ell e^{i\Omega t}(1 + e^{i\theta_2})$$

$$\dot{r}_1 = i\Omega \ell e^{i\Omega t}(1 + e^{i\theta_1}) + \ell e^{i\Omega t} i\dot{\theta}_1 e^{i\theta_1}$$

$$\dot{r}_1 = i\ell e^{i\Omega t}\left(\Omega(1 + e^{i\theta_1}) + \dot{\theta}_1 e^{i\theta_1}\right)$$

$$\dot{r}_1 = i\ell e^{i\Omega t}\left(\Omega + (\dot{\theta}_1 + \Omega) e^{i\theta_1}\right)$$

$$\dot{r}_2 = -i\ell e^{i\Omega t}\left(\Omega + (\dot{\theta}_2 + \Omega)e^{i\theta_2}\right)$$

$$V = mgh_1 + mgh_2$$

where
$h_1 = \text{Imag}(r_1)$
$h_2 = \text{Imag}(r_2)$

$$h_1 = \ell\cos\Omega t + \ell\cos(\Omega t + \theta_1)$$

$$h_2 = -\ell\cos\Omega t - \ell\cos(\Omega t + \theta_2)$$

$$\therefore \quad V = mg\ell\left(\cos(\Omega t + \theta_1) - \cos(\Omega t + \theta_2)\right)$$

$$\frac{\partial V}{\partial \theta_1} = -mg\ell\sin(\theta_1 + \Omega t)$$

$$\frac{\partial V}{\partial \theta_2} = +mg\ell\sin(\theta_2 + \Omega t)$$

Now for the ~~e~~ Kinetic Energy

$$v_1^2 = \dot{r}_1 \dot{r}_1^* = \ell^2 \left(\Omega + (\Omega + \dot{\theta}_1) e^{i\theta_1} \right) \left(\Omega + (\Omega + \dot{\theta}_1) e^{-i\theta_1} \right)$$

$$= \ell^2 \left\{ \Omega^2 + (\Omega + \dot{\theta}_1)\Omega (e^{i\theta_1} + e^{-i\theta_1}) + (\Omega + \dot{\theta}_1)^2 \right\}$$

$$v_1^2 = \ell^2 \left\{ 2\Omega^2 + 2\Omega\dot{\theta}_1 + 2\Omega(\Omega + \dot{\theta}_1)\cos\theta_1 + \dot{\theta}_1^2 \right\}$$

$$v_2^2 = \ell^2 \left\{ 2\Omega^2 + 2\Omega\dot{\theta}_2 + 2\Omega(\Omega + \dot{\theta}_2)\cos\theta_2 + \dot{\theta}_2^2 \right\}$$

$\therefore$

$$T = \frac{\ell^2}{2} m \left\{ 4\Omega^2 + 2\Omega(\dot{\theta}_1 + \dot{\theta}_2) \cancel{+ 4\Omega^2 \cos\theta} + 2\Omega^2(\cos\theta_1 + \cos\theta_2) + 2\Omega(\dot{\theta}_1\cos\theta_1 + \dot{\theta}_2\cos\theta_2) + \dot{\theta}_1^2 + \dot{\theta}_2^2 \right\}$$

The term $\frac{\ell^2}{2} m (4\Omega^2)$ will make no contribution to $\frac{\partial L}{\partial \dot{\theta}_k}$ or $\frac{\partial L}{\partial \theta}$ hence we ignore it.

$$L = T - V$$

$$= m\Omega l^2 (\dot\theta_1 + \dot\theta_2)$$

$$+ m\Omega^2 l^2 (\cos\theta_1 + \cos\theta_2)$$

$$+ ml^2\Omega (\dot\theta_1 \cos\theta_1 + \dot\theta_2 \cos\theta_2)$$

$$+ \tfrac{1}{2} ml^2 (\dot\theta_1^2 + \dot\theta_2^2)$$

$$- mgl (\cos(\Omega t + \theta_1) - \cos(\Omega t + \theta_2))$$

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or with $m = l = g = 1$

as Ω is $[\Omega] = \frac{1}{T}$

$T\Omega$ is dimensionless. i.e.

$$\Omega \to \sqrt{\frac{l}{g}}\,\Omega = \overline{\Omega}$$

$$L = \dot{\theta}_1 + \dot{\theta}_2 + (\cos\theta_1 + \cos\theta_2)\bar{\Omega}^2$$
$$+ (\dot{\theta}_1 \cos\theta_1 + \dot{\theta}_2 \cos\theta_2)\bar{\Omega}$$
$$+ \tfrac{1}{2}(\dot{\theta}_1^2 + \dot{\theta}_2^2)$$
$$- \cos(\bar{\Omega} t + \theta_1) + \cos(\bar{\Omega} t + \theta_2)$$

$$\frac{\partial \dot{\theta}_1}{\partial \dot{\theta}_1} = 1 \, , \quad \frac{\partial \dot{\theta}_2}{\partial \dot{\theta}_2} = 1$$ no contribution

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$$\therefore \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = \frac{d}{dt}\left(\bar{\Omega}(\cos\theta_1) + \dot{\theta}_1\right) = -\bar{\Omega}\dot{\theta}_1 \sin\theta_1 + \ddot{\theta}_1$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = \frac{d}{dt}\left(\bar{\Omega}(\cos\theta_2) + \dot{\theta}_2\right) = -\bar{\Omega}\dot{\theta}_2 \sin\theta_2 + \ddot{\theta}_2$$

$$\frac{\partial L}{\partial \theta_1} = -\bar{\Omega}^2 \sin\theta_1 - \dot{\theta}_1 \Omega \sin\theta_1 + \sin(\bar{\Omega} t + \theta_1)$$

$$\frac{\partial L}{\partial \theta_2} = -\bar{\Omega}^2 \sin\theta_2 - \dot{\theta}_2 \Omega \sin\theta_2 - s(\bar{\Omega} t + \theta_2)$$

Therefore the equations of motion are.

$$\ddot{\theta}_1 - \bar{\Omega}\,\dot{\theta}_1 \sin\theta_1 + \bar{\Omega}^2 \sin\theta_1$$
$$+ \dot{\theta}_1 \Omega \sin\theta_1 - \sin(\bar{\Omega}t + \theta_1) = 0$$

$$\ddot{\theta}_2 - \bar{\Omega}\,\dot{\theta}_2 \sin\theta_2 + \bar{\Omega}^2 \sin\theta_2$$
$$+ \dot{\theta}_2 \Omega \sin\theta_2 + \sin(\bar{\Omega}t + \theta_2) = 0$$

The equations for θ_1 and θ_2 are independent of each other!
why?