Flatland Mechanics

Mechanics discribes a virtual world, which to a large extent is very representative of what we call 'the real world'. In this course we shall restrict our study to the world of flatland - that is the mechanics of two dimensions. The truth is that much of what is done in basic mechanics is restricted to flatland, so little is lost in this simplification and we can concentrate on principles and avoid computational difficulties!

Mechanics starts with 'Kinematics' which is just the geometry of all possible continous motions. To describe geometry of motion we need the ideas of direction as well as magnitude. Complex numbers provide an effective and computationaly efficient tool for this purpose. So first off here is a quick review of complex arithmetic with a geometric flavour:

Recall, that we introduce $i = \sqrt{-1}$ so that an albebra equation $x^2 + 1 = 0$ has a solution $- \; x = \pm\sqrt{-1} = \pm i$.
Complex numbers are defined as numbers of the form:

$$c = a + ib$$

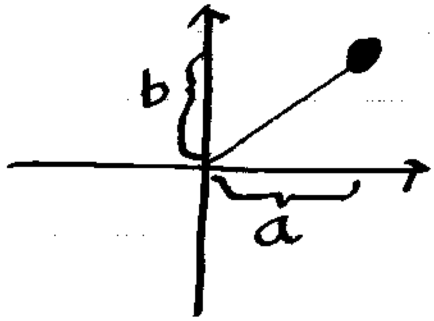
<u>with</u>

Real $c = a$
Imaginary $c = b$

We also define c^* = complex conjigate of c by changing $i \to -i$ so

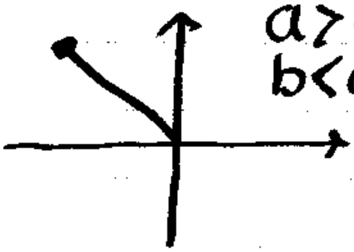
$$c^* = a - ib$$

The important interpretation of complex numbers for us is that we can take the real and imaginary parts as distance along a set of perpendicular axis, thus



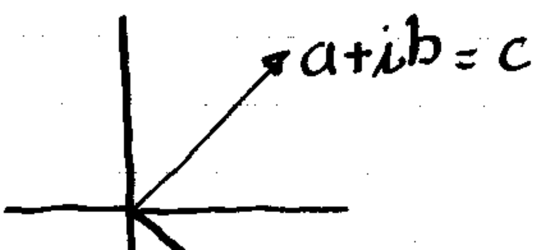
$a, b > 0$

or

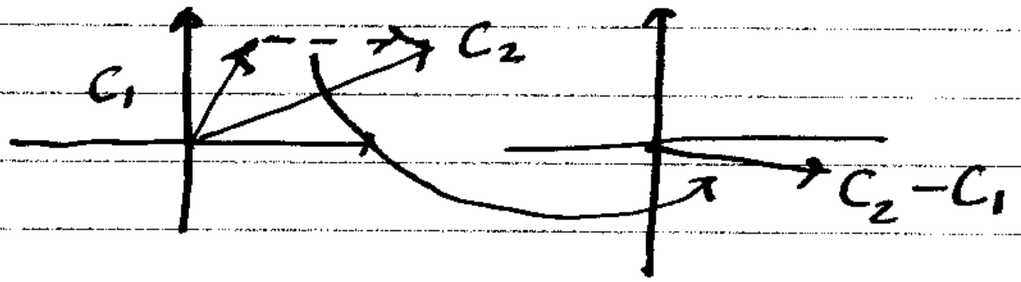


$a > 0$
$b < 0$

Thus the number also has a direction



$c_1 = a_1 + ib_1$, $c_2 = a_2 + ib_2$



and in a similiar manner we can interpret $\alpha_1 c_1 + \alpha_2 c_2$ where α_1 and α_2 are real numbers. If you are familiar with vectors you may see the correspondance with flatland vectors.

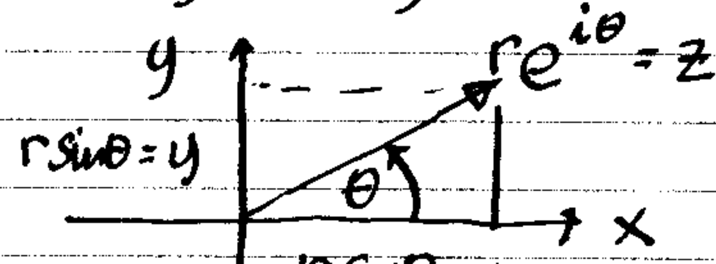
Now suppose we want to describe the motion of a particle on the circumfrence of a circle as a function of time. Euler showed that the important equation below is true.

$$z = x + iy = re^{i\theta}$$

and that $\quad r^2 = x^2 + y^2$

$$\tan\theta = y/x$$

with the geometry



Therefore the equation of a circle is that $r = \text{Constant} = r_0$ so

$$z = r_0 e^{i\theta(t)}$$

where we take θ to be a function of time 't'.
A simple case is that the particle moves with a constant <u>angular velocity</u> so

$$\theta = \omega_0 t$$

$$\frac{d\theta}{dt} = \dot{\theta} = \omega_0$$

$$z = r_0 e^{i\omega_0 t}$$

We define the <u>complex velocity</u> as

$$V = \dot{z} = \frac{dz}{dt}$$

So in this case we have.

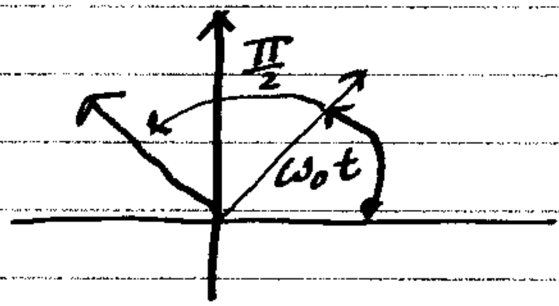
$$\dot{z} = \frac{d}{dt} r_0 e^{i\theta(t)} = \frac{d}{dt} r_0 e^{i\omega_0 t}$$

$$= i\omega_0 r_0 e^{i\omega_0 t}$$

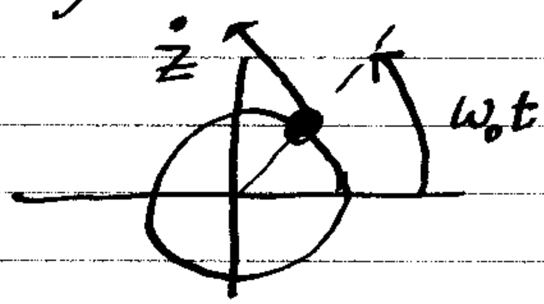
Now $i = 0 + 1i = \cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) i$

$$= e^{i\frac{\pi}{2}}$$

and $\dot{z} = \omega_o r_o e^{i\frac{\pi}{2}} e^{i\omega_o t}$
$$= \omega_o r_o e^{i(\omega_o t + \frac{\pi}{2})}$$



which show that the velocity is at right angles to the position direction – i.e. tangent to the circle!



The magnitude of $z = |z| = \sqrt{x^2+y^2} = r$

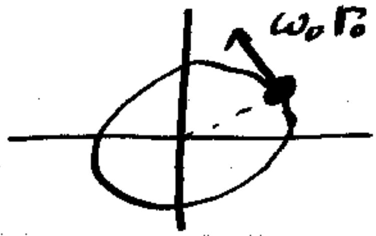
and in general $|z|^2 = z^* z$
$= (x-iy)(x+iy) = x^2 - iyx + ixy - i^2 y^2$
$= x^2 + y^2 = r^2$ so

$$|z| = \sqrt{z^* z}$$

So in our example

$$|V|^2 = \dot{z}^* \dot{z} = (-i\omega_o r_o e^{-i\omega_o t})(i\omega_o r_o e^{i\omega_o t})$$
$$= \omega_o^2 r_o^2$$

So the velocity of the particle is perpendicular or orthogonal to the position direction with magnitude (speed) $\omega_0 r_0$



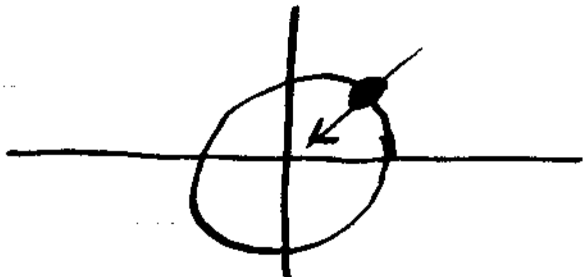
Further we define the acceleration as

$$A = \frac{d^2 z}{dt^2}$$ so in our example:

$$A = \frac{d^2}{dt^2} r_0 e^{i\omega_0 t} = \frac{d}{dt} i\omega_0 r_0 e^{i\omega_0 t}$$

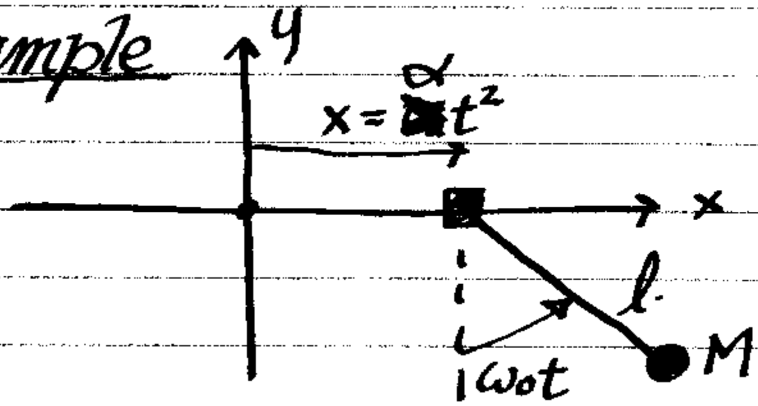
$$A = -\omega_0^2 r_0 e^{i\omega_0 t}$$

The direction is now along the position direction but toward the origin! The magnitude is $+\omega_0^2 r_0$.



This is sometimes called the centripetal acceleration.

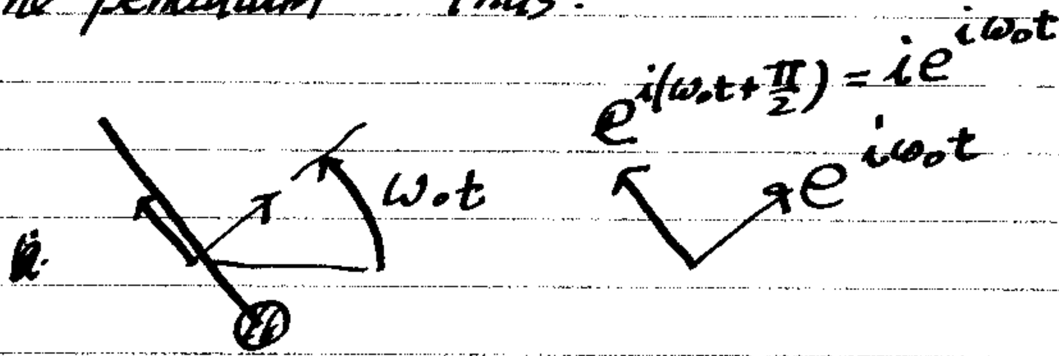
<u>example</u>



A slider moves in the positive x direction with speed αt^2. A pendulum of length l rotates about the slider with angular position given by $\omega_0 t$ from the vertical as shown. Find the velocity and acceleration of M.

Before attacking this problem note that we can represent a pure direction as $e^{i\phi}$ and that $|e^{i\phi}| = \sqrt{e^{-i\phi}e^{i\phi}} = \sqrt{1} = 1$ —
~~Thus the direction of two orthogonal~~

let's place an orthogonal set of directions along the pendulum — thus.



With this it is eazy to see that the position M is given by:

$$Z = \alpha t^2 - li e^{i\omega_0 t}$$

hence

$$V = \dot{Z} = 2\alpha t - li\,(i\omega_0)\, e^{i\omega_0 t}$$

$$V = 2\alpha t + \omega_0 l e^{i\omega_0 t}$$

$$V = |V| e^{i\phi}$$

where as

$$V = (2\alpha t + \omega_0 l \cos\omega_0 t) + i\,\omega_0 l \sin\omega_0 t$$

$$|V| = \sqrt{(2\alpha t + \omega_0 l \cos\omega_0 t)^2 + (\omega_0 l \sin\omega_0 t)^2}$$

$$\phi = \tan^{-1}\left(\frac{\omega_0 l \sin\omega_0 t}{2\alpha t + \omega_0 l \cos\omega_0 t}\right)$$

The acceleration A is

$$A = \dot{V} = 2\alpha + i\omega_0^2 l e^{i\omega_0 t}$$

and the reader can compute $|A|$, ψ in

$$A = |A| e^{i\psi}$$

We now show that given two complex numbers w, v

~~if $\perp$ $w^* v$~~

$$\mathrm{Real}(w^* v) = \mathrm{Real}(v^* w) = 0$$

implies either w or $v = 0$
or $w \perp v$.

$$\mathrm{Imag.}(w^* v) = \mathrm{Imag}(w v^*) = 0$$

implies either w or $v = 0$
or $w \parallel v$

---

let $w = w e^{i\psi}$, $v = v e^{i\phi}$

$$w^* v = wv\, e^{i(-\psi + \phi)}$$

if $\phi - \psi = \pm\frac{\pi}{2}$

$$e^{i(\phi-\psi)} = e^{\pm i\frac{\pi}{2}} = \cos\frac{\pi}{2} \pm i\sin\frac{\pi}{2}$$

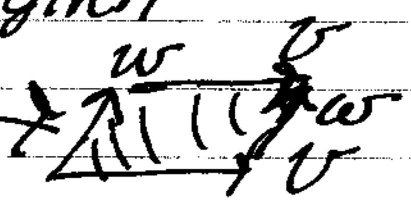
$\cos\frac{\pi}{2} = 0$,

also note addition of $n\pi$, $n=\pm1,\pm2\cdots$ etc. does not change this.

∴ if $w \perp v$ $\mathrm{Real}(w^*v)=0$

We leave it as an exersize to show the parallel case.

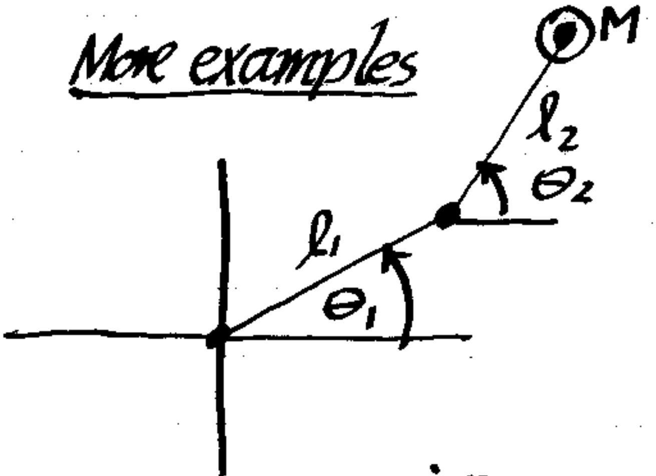
Also given



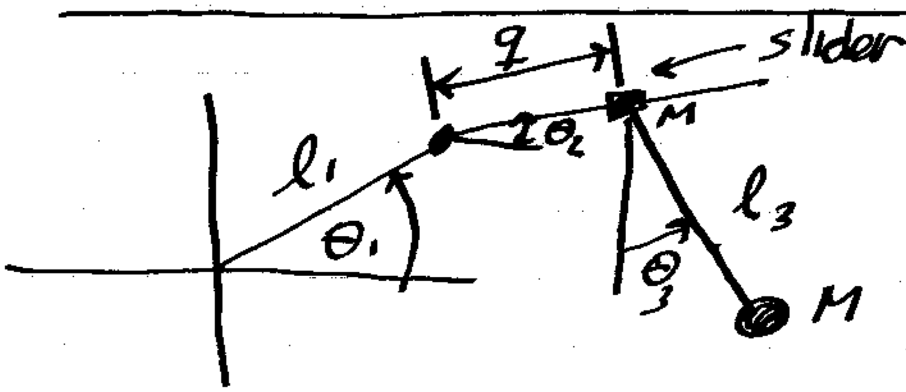
show that $|\mathrm{Im}(w^*v)| = |\mathrm{Im}(v^*w)|$

= area surrounded by w, v as indicated.

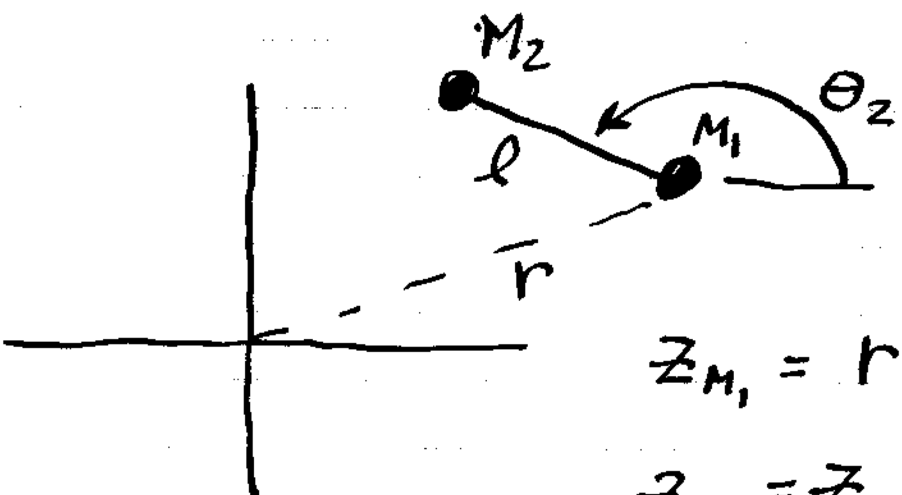
More examples



$$z_M = l_1 e^{i\theta_1} + l_2 e^{i\theta_2}$$



$$z_M = l_1 e^{i\theta_1} + \underline{q} e^{i\theta_2} - i l_3 e^{i\theta_3}$$



$$z_{M_1} = r e^{i\theta} = x + iy$$

$$z_{M_2} = z_{M_1} + l e^{i\theta_2}$$

<u>Kinematics</u> - <u>all possible continious motion.</u>

<u>Dynamics</u> - <u>introduce</u> Mass of Particle m
<u>introduce</u> Force F

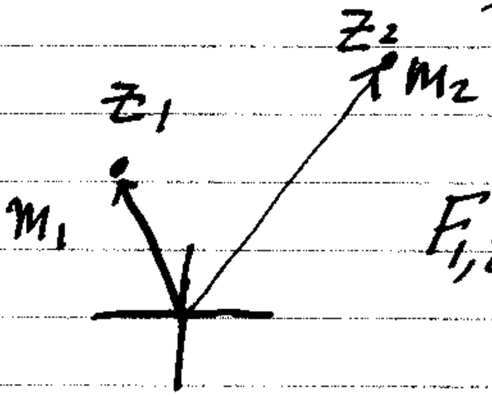
Motion must obey <u>Newton's Equations</u>

$$F = m\ddot{z}$$

<u>Free particles</u>

Active Forces , $F =$:

examples $F_{1,2}$ = Force on particle 1
from particle 2



$$F_{1,2} = G\frac{m_1 m_2}{|z_1 - z_2|^2}\frac{z_2 - z_1}{|z_2 - z_1|}$$

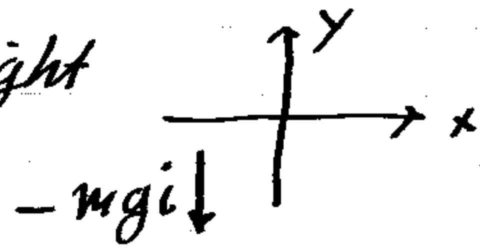
$$F_{2,1} = G\frac{m_1 m_2}{|z_1 - z_2|^2}\frac{z_1 - z_2}{|z_1 - z_2|}$$

$$F_{1,2} + F_{2,1} = 0$$

$$G = 6.67 \times 10^{-11}\left(\frac{\text{meters}^3}{\text{sec}^2\ \text{Kg.}}\right)$$

flat earth in a flatland
with gravitational acceleration $g \cong 10\ m/sec^2$

$F = -mgi$ = weight



Viscous resistance

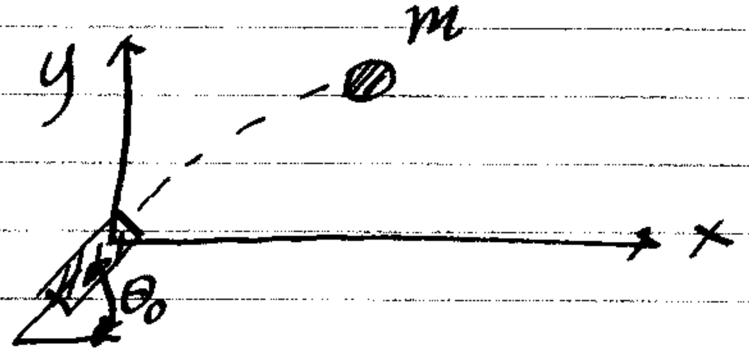
$$\bigcirc \longrightarrow \dot{z}$$

$$F = -\nu \dot{z}$$

'Wind' Resistance

$$F = -\frac{\dot{z}}{|\dot{z}|}(z^* z) = -\dot{z}\,|\dot{z}|$$

Projectile Motion



The inital position $z = 0$
The inital velocity $\dot{z} = V_0 = v_0 e^{i\theta_0}$
gravitational force $F = -img$.

$\therefore \; m\ddot{z} = -img$

$\ddot{z} = -ig.$

$\therefore \; z = -\frac{igt^2}{2} -igtC + D$

check $\dot{z} = V = -igt - igC$

$\ddot{z} = A = -ig$ ✓

Apply initial conditions at $t=0$

$0 = D$

$V_0 = v_0 e^{i\theta_0} = -igC$

$\therefore \; C = \frac{v_0 e^{i\theta_0}}{(-ig)} = i\frac{v_0 e^{i\theta_0}}{g}.$

hence:

$$z = -\frac{igt^2}{2} - igt\left(\frac{iv_0 e^{i\theta_0}}{g}\right)$$

$$\boxed{z = v_0 t e^{i\theta_0} - i\frac{1}{2}gt^2}$$

$$x = v_0 t \cos\theta_0, \quad y = -\frac{1}{2}gt^2 + v_0 t \sin\theta_0$$

as $\quad t = \dfrac{x}{v_0 \cos\theta_0}$

$$y = -\frac{1}{2}g \cdot \frac{x^2}{v_0^2\cos^2\theta_0} + v_0 t \sin\theta_0$$

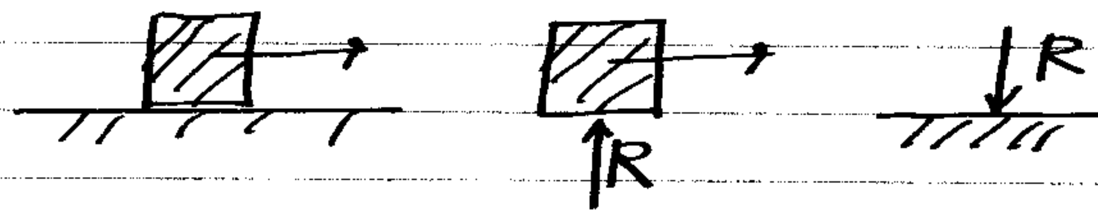
$$y = -\frac{1}{2}\frac{g}{v_0^2\cos^2\theta_0}x^2 + \frac{v_0 \sin\theta_0 \, x}{v_0\cos\theta_0}$$

$$y = -\frac{1}{2}\frac{g}{v_0^2\cos^2\theta_0}x^2 + (\tan\theta_0)\,x$$

$$y = x\left(\tan\theta_0 - \frac{1}{2}\frac{g}{v_0^2\cos\theta_0}x\right)$$

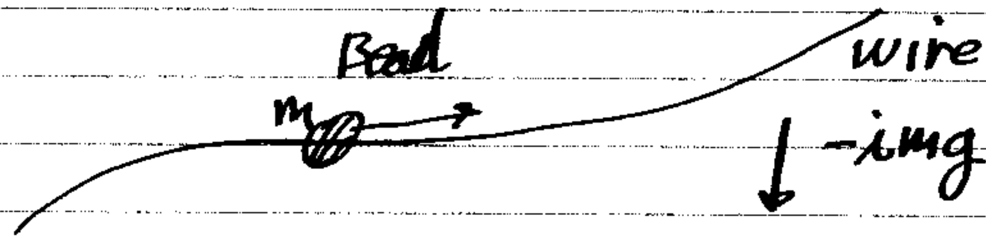
$\therefore$ max range $\quad x_{max} = \left(\dfrac{2v_0^2\cos\theta_0}{g\tan\theta_0}\right)$!

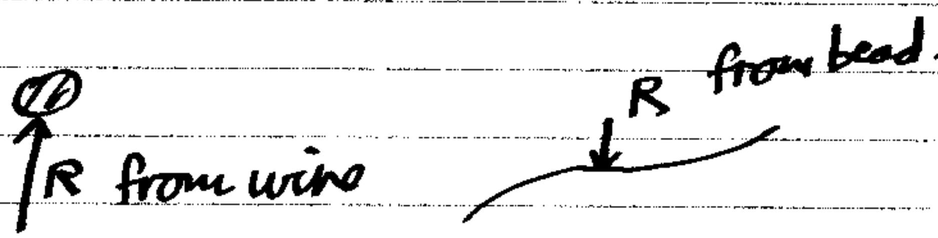
Constraint Forces



Newtons Equations are incomplete!

Constraint Forces are unknown.





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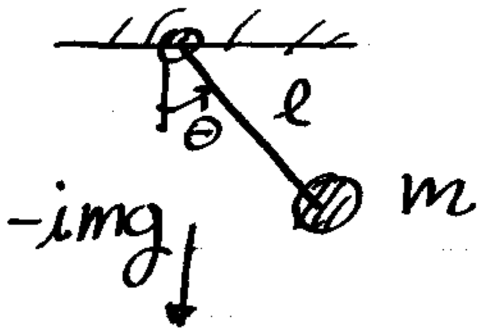
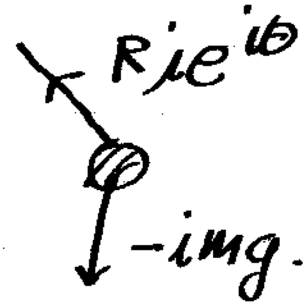
example

simple pendulum.

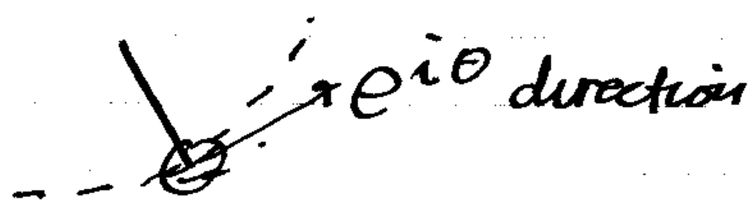




$$m\ddot{z} = -img + Rie^{i\theta}$$

$$z = l(-ie^{i\theta})$$

$$\dot{z} = V = -il(i\dot{\theta}e^{i\theta}) = l\dot{\theta}e^{i\theta}$$



$$\ddot{z} = \dot{V} = \underbrace{l\ddot{\theta}e^{i\theta}}_{\text{(radial)}} + \underbrace{il\dot{\theta}^2e^{i\theta}}_{\text{(tangential)}}$$

$$\therefore \cancel{z}\; m(l\ddot{\theta}e^{i\theta} + il\dot{\theta}^2e^{i\theta}) = -img + Rie^{i\theta}$$

Since the constraint force is ~~tangen~~ radial (perpendicular to V) multiplication by $e^{-i\theta}$ will eliminate it from equations.

Therefore Multiply by $e^{-i\theta}$ and take real part!

$$ml\ddot{\theta} + iml\dot{\theta}^2 = -imge^{-i\theta} + iR$$

seperating real and imag parts.

$$ml\ddot{\theta} = -mg\sin\theta$$

$$ml\dot{\theta}^2 = -mg\cos\theta + R$$

$\therefore$ equation of motion

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

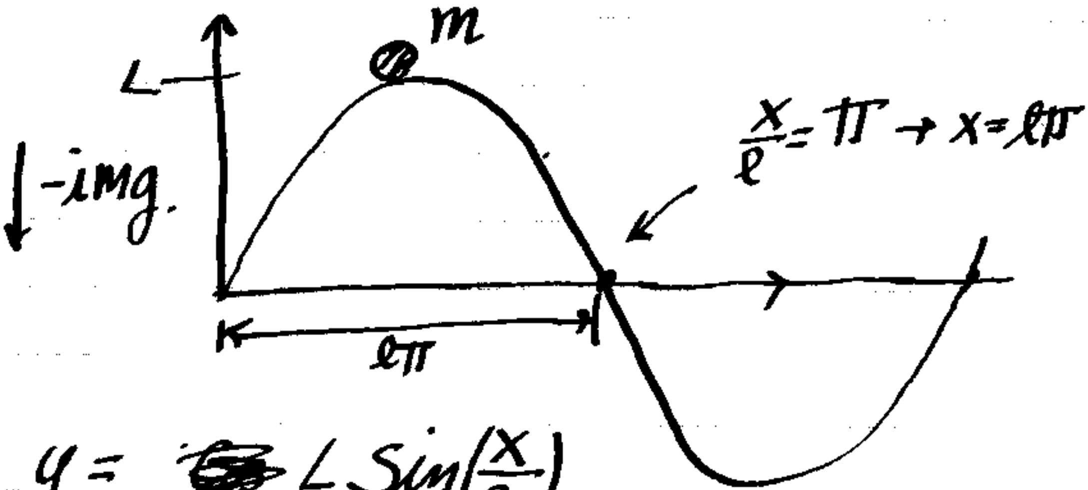
equation of constraint.

$$R = \underset{\uparrow}{ml\dot{\theta}^2} + mg\cos\theta$$

Centripital ~~centrifugal~~ force on particle

Note: to find the constraint force we must first find $\theta(t)$.

Roller Coaster Mechanics



$$y = \cancel{}\, L \sin\left(\frac{x}{l}\right)$$

$$\therefore \quad Z = x + i L \sin\left(\frac{x}{l}\right)$$

$$V = \dot{x} + i \frac{\dot{x} L}{l} \cos\left(\frac{x}{l}\right)$$

$$\cancel{A = \ddot{x} + i \frac{\ddot{x} L}{L} \frac{x}{l} =}$$

$$A = \ddot{x} + i \frac{\ddot{x} L}{l} \cos\frac{x}{l} - i \frac{\dot{x} L}{l} \frac{\dot{x}}{l} \sin\frac{x}{l}$$

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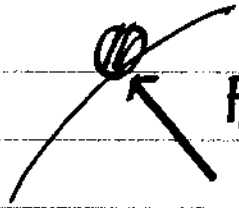
hence

$$m\left(\ddot{x} + i \frac{\ddot{x} L}{l} \cos\frac{x}{l} - i \frac{\dot{x}^2}{l^2} L \sin\frac{x}{l}\right)$$

$$= -img + F_{constraint}.$$



$F_{constraint} \perp$ to V

or $\perp$ to $\left[1 + i\frac{L}{\ell}\cos\left(\frac{x}{\ell}\right)\right]$

$$\therefore \boxed{\begin{aligned} F_{cons.} &= Ri\left[1 + i\frac{L}{\ell}\cos\frac{x}{\ell}\right] \\ &= -R\left[\frac{L}{\ell}\cos\frac{x}{\ell} - i\right] \end{aligned}}$$

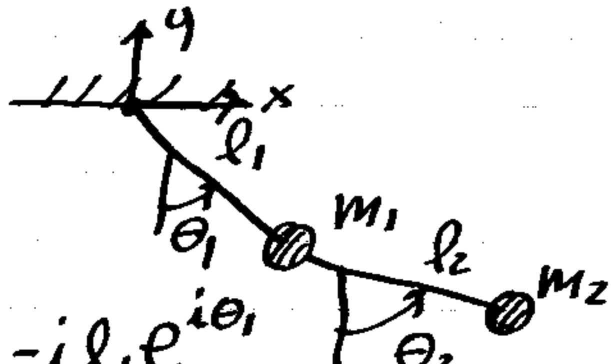
$$\boxed{\begin{aligned} m\left\{\ddot{x} + i\frac{\ddot{x}L}{\ell}\cos\frac{x}{\ell} - i\frac{\dot{x}^2}{\ell^2}L\sin\frac{x}{\ell}\right\} \\ = -img - R\left[\frac{L}{\ell}\cos\frac{x}{\ell} - i\right] \end{aligned}}$$

Multiply by $\left[1 - i\frac{L}{\ell}\cos\frac{x}{\ell}\right]$ to obtain constraint free equation of motion! (The real part)

The imaginary part will provide an equation for R.

Multi particle systems

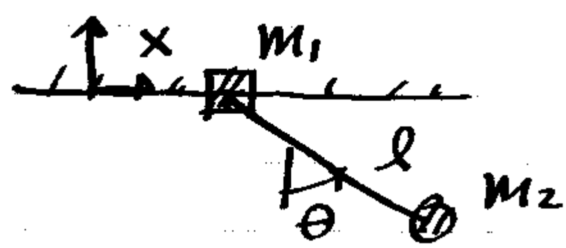
To describe a system of many particles we introduce the Multi-Position matrix. e.g. for a double pendulum



$$z_1 = -i l_1 e^{i\theta_1}$$
$$z_2 = -i l_1 e^{i\theta_1} - i l_2 e^{i\theta_2}$$

$$\mathbb{Z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -i l_1 e^{i\theta_1} \\ -i l_1 e^{i\theta_1} - i l_2 e^{i\theta_2} \end{bmatrix}$$

For a pendulum with a sliding support.



$$z_1 = x$$
$$z_2 = x - i l e^{i\theta}$$

$$\mathbb{Z} = \begin{bmatrix} x \\ x - i l e^{i\theta} \end{bmatrix}$$

In the same manner we introduce the multi Velocity ~~Vector~~ Matrix.

$$\mathbb{V} = \frac{d\,\mathbb{Z}}{dt}$$

e.g. $$\mathbb{V} = \begin{bmatrix} \dot{x} \\ \dot{x} - i l i \dot{\theta} e^{i\theta} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{x} + l \dot{\theta} e^{i\theta} \end{bmatrix}$$

We can extract velocity modes out of this as follows.

Define the Kinematic Differential Eqs.

$$\left.\begin{aligned} \dot{x} &= v \\ \dot{\theta} &= \omega \end{aligned}\right\}$$ these define the speed variables v, ω

let the Direction Matrices Be defined as

$$\mathbb{Q}_v = \frac{\partial \mathbb{V}}{\partial v}, \quad \mathbb{Q}_\omega = \frac{\partial \mathbb{V}}{\partial \omega}$$

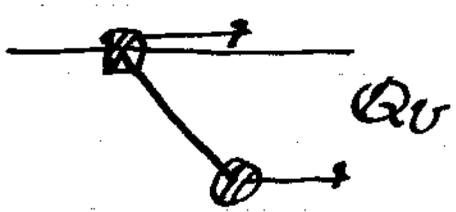
Note: we must first substitute v and ω for $\dot{x}, \dot{\theta}$

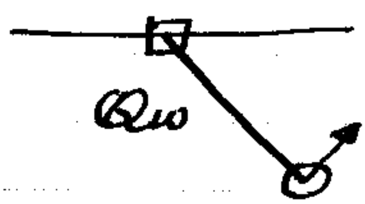
$$\mathbb{V} = \begin{bmatrix} v \\ v + l\omega e^{i\theta} \end{bmatrix}$$

$$Q_v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad Q_\omega = \begin{bmatrix} 0 \\ l e^{i\theta} \end{bmatrix}$$

Q_v and Q_ω define directions for allowed motions. Any general motion is a linear sum of these — i.e.

$$\mathbb{V} = v\,Q_v + \omega\,Q_\omega$$

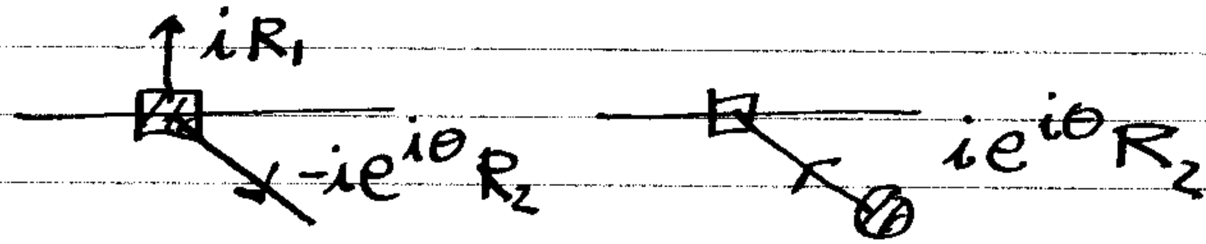




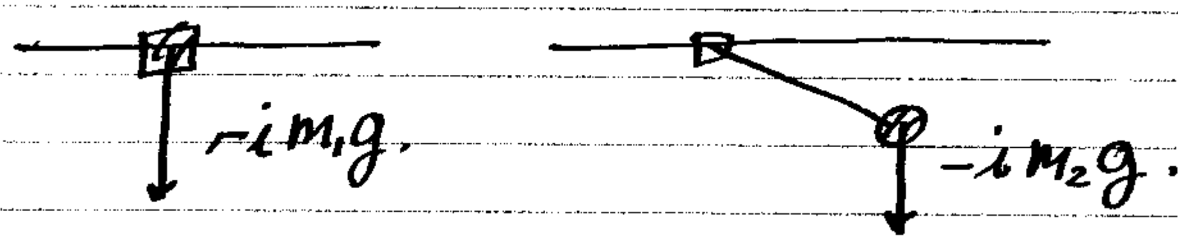
Now we can calculate the <u>Momentum Matrix</u>

$$\mathbb{P} = \begin{bmatrix} m_1 v \\ m_2 v + m_2 l\omega e^{i\theta} \end{bmatrix}$$

Constraint Forces



Applied forces



The Force Matrix is

$$\mathbb{F} = \begin{bmatrix} -im_1g. - ie^{i\theta}R_2 + iR_1 \\ \\ -im_2g. + ie^{i\theta}R_2 \end{bmatrix}$$

Newtons Equations are thus.

$$\dot{\mathbb{P}} = \mathbb{F}$$

Now define $\mathbb{Q}^* = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}^* = (Q_1^*, Q_2^*)$

i.e. $\mathbb{Q}^*$ is the complex conjigate transpose of the column $\mathbb{Q}$ - hence a row. this is sometimes called the 'hermetian conjigate.' Now.

$$\mathbb{Q}_j^*(\dot{\mathbb{P}} - \mathbb{F}) = 0$$

where $\mathbb{Q}_j = \mathbb{Q}_v$ or $\mathbb{Q}_w$

We will see that the real part of this is free of constraint forces and gives us the equations of motion free of constraint force. The imaginary part gives us equations for R_1 and R_2.

What is important is that if we ignore the constraint forces $\mathbb{F} = \mathbb{F}_a + \mathbb{F}_{constr.}$

$$\text{Real}\left(\mathbb{Q}_j^* \cdot (\dot{\mathbb{P}} - \mathbb{F}_a)\right) = 0$$

will give the correct equations of motion! One equation for each $\mathbb{Q}_j$.

Together with the kinematic differential equations we get a set of first order differential equations. In our example this gives:

$$\left.\begin{aligned} \text{Real } \mathbb{Q}_v^* (\dot{\mathbb{P}} - \mathbb{F}_a) &= 0 \\ \text{Real } \mathbb{Q}_\omega^* (\dot{\mathbb{P}} - \mathbb{F}_a) &= 0 \\ \dot{v} &= x \\ \dot{\omega} &= \theta \end{aligned}\right] \quad \begin{array}{l} \text{dependent variables} \\ x, \theta, v, \omega \end{array}$$

Of course we must provide 4 initial conditions for x, θ, v and ω.

<u>Now</u>

$$\dot{\mathbb{P}} = \begin{bmatrix} m_1 \dot{v} \\ m_2 \dot{v} + m_2 l \dot{\omega} e^{i\theta} + i m_2 l \omega^2 e^{i\theta} \end{bmatrix}$$

$$Q_v^* \dot{\mathbb{P}} = (1, 1)\, \dot{\mathbb{P}}$$

$$= (m_1 + m_2)\dot{v} + m_2 l \dot{\omega} e^{i\theta} + i m_2 l \omega^2 e^{i\theta}$$

$$Q_\omega^* \dot{\mathbb{P}} = (0, l e^{-i\theta})\, \dot{\mathbb{P}}$$

$$= m_2 l e^{-i\theta} \dot{v} + m_2 l^2 \dot{\omega} + i m_2 l^2 \omega^2$$

$$Q_v^* \mathbb{F} = (1,1)\mathbb{F} = -i(m_1 + m_2) g + i R_1$$

$$Q_\omega^* \mathbb{F} = (0, l e^{-i\theta})\mathbb{F}$$

$$= l e^{-i\theta}(-i m_2 g) + i l R_2$$

taking real parts (note R_1 and R_2 are now gone)

With $m_1 + m_2 = m$

$$m\dot{v} + m_2 l \dot{\omega} \cos\theta - m_2 l \omega^2 \sin\theta = 0$$

$$(m_2 l \cos\theta)\dot{v} + m_2 l^2 \dot{\omega} = -l m_2 g \sin\theta$$

and the kinematic differential equations

$$\dot{\theta} = \omega$$
$$\dot{x} = v$$

Note $m_1 \to \infty \to \dot{v} = 0$

$$\to \quad \dot{\omega} + \frac{g}{l} \sin\theta = 0$$

$$\ddot{\theta} + \frac{g}{l} \sin\theta = 0$$

as one would expect.

We leave it to the reader to develop the expressions for R_1, R_2.

The important point is that constraint free equations are given by

$$\boxed{\mathrm{Real}\left(\mathbb{Q}_{j\cdot}^{*}\left(\dot{\mathbb{P}} - \mathbb{F}_a\right)\right) = 0} \quad (+)$$

where for 'coordinates' q_j

with $\dot{q}_{j\cdot} = v_{j\cdot}$

$$\mathbb{Q}_{j\cdot} = \frac{\partial \mathbb{V}}{\partial q_{j\cdot}}$$

(+) are sometimes called Kane's Equations

the components of $\mathbb{Q}_{j\cdot}$ are called partial velocities, the $\dot{q}_{j\cdot}$ are generalized speeds, and

$\mathbb{Q}_j^* \mathbb{F}_a$ is the jth generalized applied force

$-\mathbb{Q}_j^* \dot{\mathbb{P}}$ is the jth generalized inertia force.

Another generalization is that we may use kinematic differential equations of the form

$$\dot{q}_j = \sum_{i=1}^{n} q_i C_{ij}(q) + D_j(q)$$

however we will not use this in the present course.
