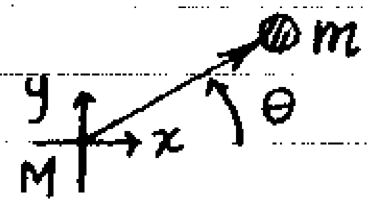
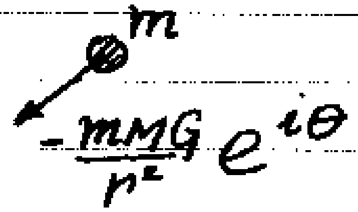
The Kepler Problem





The origin of the force is taken as the origin of the basic reference frame! The justification for this is that $M >> m$ - for example the mass of the sun is far greater then the mass of any planet.

Newton's second law gives us (m drops out)

$$\boxed{\ddot{z} = -\frac{k}{r^2}e^{i\theta}} \qquad \boxed{k = GM}$$

here $[k] = [\ddot{z}r^2] = L^3T^{-2}$

$$[G] = L^3T^{-2}M^{-1}$$

in SI units $G \approx 6.67 \times 10^{-11}$ meters^3/Kg. sec^2

we take d as a characteristic length e.g. the approx radius of an orbit —
Then $[k/d] = L^2T^{-2}$
That is k has the dimensions of a velocity squared — hence we take

$$v_c^2 = K/d \quad \text{or} \quad v_c = \sqrt{K/d}$$

The Mass of the Sun is $1.99 \cdot 10^{30}$ Kg.
Earth Sun distance (A.U.) $1.49 \cdot 10^{8}$ Km
$= 1.49 \cdot 10^{11}$ meters

$$GM_{sun} = K_{sun} = 6.67 \cdot 10^{-11} \times 1.99 \cdot 10^{30}$$
$$\simeq 13 \cdot 10^{19}$$

$$\therefore \quad K/d \simeq \frac{13 \cdot 10^{19}}{1.49 \cdot 10^{11}} \simeq 9 \cdot 10^{8}\ m^2/s^2$$

$$\therefore \quad v_c \simeq 3 \cdot 10^{4}\ m/s \doteq 30\ Km/s$$

Using the Circumfrance of the earths orbit about the sun ~~and is~~ $2\pi d$
$v \simeq 2\pi d / (365 \times 24 \times 60 \times 60)$ sec.
$\simeq 3$[illegible] Km/s.

You can estimate the orbital velocity of the other planets using Bode-Titius empirical law:

take	→ 0	3	6	12	24	48	...
+4	→ 4	7	10	16	28	52	...
÷10	→ 0.4	0.7	1.0	1.6	2.8	5.2	A.U.
	mercury	venus	earth	mars	"Asteroid Belt"	Jupiter	

Not too accurate for Neptune — very off for Pluto.

As another useful parameter we take
v_m = fastest orbital speed
if $r = d$ at this speed we have
$\dot{\theta} = \dot{\theta}_m$ when $r = d$.
$v_m = (\dot{\theta}_m d)$

Now if $z = re^{i\theta}$

$$dz/dt = \dot{z} = \dot{r}e^{i\theta} + i r\dot{\theta}e^{i\theta}$$
$$\ddot{z} = \ddot{r}e^{i\theta} + 2i\dot{r}\dot{\theta}e^{i\theta} + ir\ddot{\theta}e^{i\theta} - r\dot{\theta}^2 e^{i\theta}$$

hence

$$(\ddot{r} - r\dot{\theta}^2) + i(2\dot{r}\dot{\theta} + r\ddot{\theta}) = -\frac{k}{r^2}$$

Therefore taking the Imaginary part of this expression we have.

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

or dividing thru by $r\dot{\theta}$

$$\frac{\ddot{\theta}}{\dot{\theta}} + 2\frac{\dot{r}}{r} = 0$$

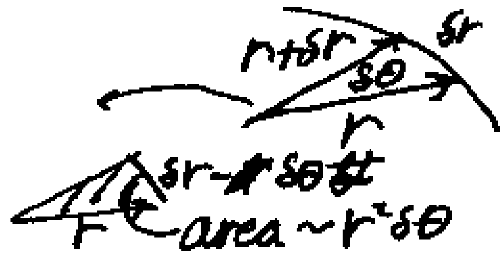
recalling that $\frac{d}{dt} \ln u(t) = \frac{\dot{u}(t)}{u(t)}$

we see that

$$\frac{d}{dt}(\ln\dot{\theta} + 2\ln r) = 0$$

or $\boxed{r^2\dot{\theta} = h}$ where h is a constant.

Note that the area velocity



$$\delta A \simeq \frac{1}{2} r^2 \delta\theta$$
$$\delta\vec{A}/\delta t \simeq \frac{1}{2} r^2\dot{\theta}$$

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which establishes <u>Keplers second law</u>

<u>The area velocity is constant</u>.

which also establishes that the motion remains planar!

Return to the equation

$$\boxed{\ddot{z} + kr^{-2}e^{i\theta} = 0}$$

as $r^2\dot{\theta} = h \Rightarrow r^{-2} = \frac{1}{h}\dot{\theta}$

$$\ddot{z} + \left(\frac{k}{h}\right)\dot{\theta} e^{i\theta} = 0$$

But $\frac{d}{dt} e^{i\theta} = i\dot{\theta} e^{i\theta}$

which means we can write this as.

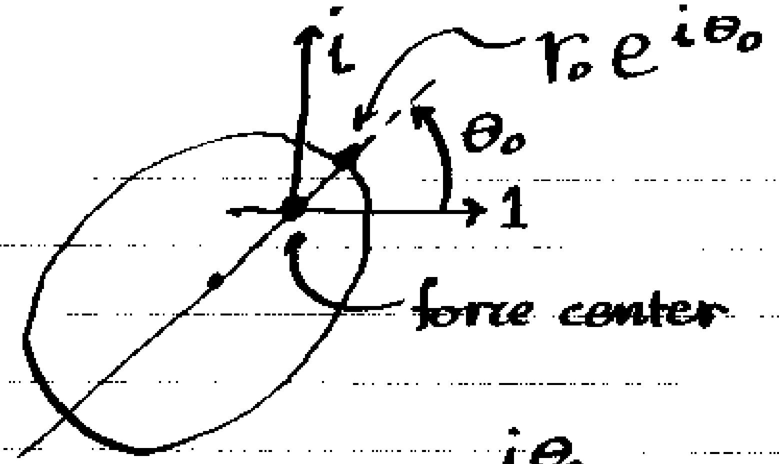
$$\frac{d}{dt}\left(\dot{z} + \frac{k}{ih} e^{i\theta}\right) = 0$$

or $\frac{d}{dt}\left(\dot{z} - \frac{k}{h} i e^{i\theta}\right) = 0$

or $\dot{z} - \frac{k}{h} i e^{i\theta} = C$

where C is a complex constant.

Assuming we have an elliptic orbit (not yet shown) let us interpret this result.



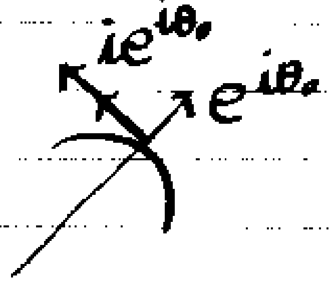
$z_0 = r_0 e^{i\theta_0}$ point of closest aproach 'para...'

$$\dot{z} = \dot{r} e^{i\theta} + i r \dot{\theta} e^{i\theta}$$

at z_0 the approach velocity $\dot{r}_0 \to 0$
Thus $r_0 = d$

$$C = \dot{z} - \frac{K}{h} i e^{i\theta} = i r_0 \dot{\theta}_0 e^{i\theta_0} - \frac{K}{h} i e^{i\theta_0}$$

$$\dot{z} - \frac{K}{h} i e^{i\theta} = \left(r_0 \dot{\theta}_0 - \frac{K}{h}\right) i e^{i\theta_0}$$



recall the parameters
$v_c^2 = K/d$, $v_m = \dot{\theta}_m d$

as the angular velocity is greatest at the point z_0 (from $r^2\dot{\theta} = h = \text{Const.}$)
$\dot{\theta}_m = \dot{\theta}_0$ and also $r_0 = d$
we have

$$r_0 \dot{\theta}_0 = v_m \quad , \quad \frac{K}{r_0} = v_c^2$$

$$\frac{K}{h} = \frac{K}{r_0^2 \dot{\theta}_0} = \frac{K}{r_0 v_m} = \frac{v_c^2}{v_m}$$

and thus

$$r_0\dot\theta_0 - \frac{k}{h} = v_m - \frac{v_c^2}{v_m} = v_m\left(1 - \frac{v_c^2}{v_m^2}\right)$$

So that

$$\dot z - \frac{v_c^2}{v_m} i e^{i\theta} = v_m\left(\frac{\dot z}{v_m} - \left(\frac{v_c^2}{v_m^2}\right) i e^{i\theta}\right)$$

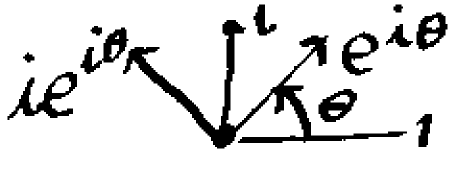
hence finally

$$\frac{\dot z}{v_m} - \frac{v_c^2}{v_m^2}\, i e^{i\theta_0} = \left(1 - \frac{v_c^2}{v_m^2}\right) i e^{i\theta_0}$$

$$v_m = r_0\dot\theta_0\ , \quad v_c = \sqrt{k/r_0}$$

r_0 = closest distance
θ_0 = angle θ at this pt. $\Big\}\ z_0 = r_0 e^{i\theta_0}$

$$\dot z = \dot r e^{i\theta} + r\dot\theta\, i e^{i\theta}$$



We can now use this to find the orbit equation for r in terms of θ.

$$\dot{z} = \dot{r}e^{i\theta} + r\dot{\theta}ie^{i\theta} \quad \text{but} \quad \dot{\theta} = h/r^2$$

$$\dot{z} = (\dot{r} + i\tfrac{h}{r})e^{i\theta} \quad \text{hence.}$$

$$\left(\frac{\dot{r}}{v_m} + i\frac{h}{r v_m} - i\frac{v_c^2}{v_m^2}\right)e^{i\theta} = \left(1 - \frac{v_c^2}{v_m^2}\right)ie^{i\theta_0}$$

$$\text{and } \frac{h}{r} = \frac{r_0^2\dot{\theta}}{r} = \frac{r_0}{r}v_m$$

$$\left[\frac{\dot{r}}{v_m} + i\left(\frac{r_0}{r} - \frac{v_c^2}{v_m^2}\right)\right]e^{i(\theta-\theta_0)} = \left(1 - \frac{v_c^2}{v_m^2}\right)i$$

for convenience let $\alpha = \dfrac{r_0}{r} - \dfrac{v_c^2}{v_m^2}$

$$\beta = 1 - v_c^2/v_m^2$$

$$\theta - \theta_0 = \Delta\theta$$

$$\left[\frac{\dot{r}}{v_m} + i\alpha\right]e^{i\Delta\theta} = i\beta$$

$\therefore$ taking. real and imag. parts.

$$\cos\Delta\theta\,\frac{\dot{r}}{v_m} - \alpha\sin\Delta\theta = 0$$

$$\alpha\cos\Delta\theta + \frac{\dot{r}}{v_m}\sin\Delta\theta = \beta$$

This is a set of linear equations for $\dot{r}/v_m$ and α so

$$\underbrace{\begin{bmatrix} \cos\Delta\theta & -\sin\Delta\theta \\ \sin\Delta\theta & \cos\Delta\theta \end{bmatrix}}_{T(\Delta\theta)}\begin{bmatrix} \dot{r}/v_m \\ \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ \beta \end{bmatrix}$$

$T(\Delta\theta)$ two d rotation → inverse is $T(-\Delta\theta)$ therefore.

$$\begin{bmatrix} \dot{r}/v_m \\ \alpha \end{bmatrix} = \begin{bmatrix} \cos\Delta\theta & \sin\Delta\theta \\ -\sin\Delta\theta & \cos\Delta\theta \end{bmatrix}\begin{bmatrix} 0 \\ \beta \end{bmatrix}$$

$$\alpha = \frac{r_0}{r} - \frac{v_c^2}{v_m^2}$$

$$= \left(1 - \frac{v_c^2}{v_m^2}\right)\cos(\theta - \theta_0)$$

$$\frac{r_0}{r} = \frac{v_c^2}{v_m^2} + \left(1 - \frac{v_c^2}{v_m^2}\right)\cos(\theta - \theta_0)$$

$$\boxed{\left(\frac{r}{r_0}\right) = \frac{1}{\frac{v_c^2}{v_m^2} + \left(1 - \frac{v_c^2}{v_m^2}\right)\cos(\theta - \theta_0)}}$$

To obtain the form given in 'Acheson' multiply both numerator & denominator by v_m^2/v_c^2

$$\boxed{\frac{r}{r_0} = \frac{v_m^2/v_c^2}{1 + \left(\frac{v_m^2}{v_c^2} - 1\right)\cos(\theta - \theta_0)}}$$

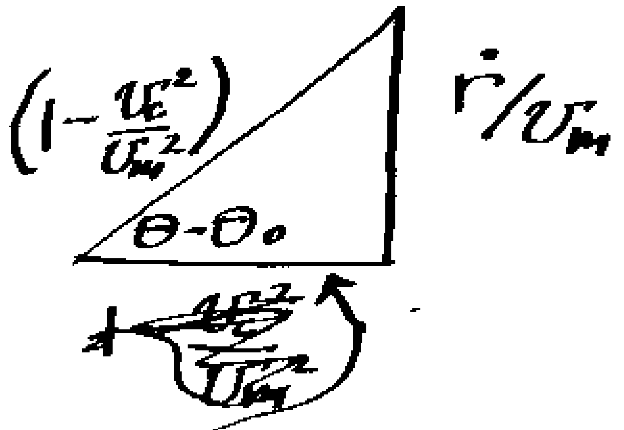
Note that our v_m is v in 'Acheson'

we also have the result

$$\frac{\dot{r}}{v_m} = \beta \sin(\theta - \theta_0) = \left(1 - \frac{v_c^2}{v_m^2}\right)\sin(\theta - \theta_0)$$

$$\left(\frac{\dot{r}}{v_m}\right) = \left(1-\frac{v_c^2}{v_m^2}\right) \sin(\theta-\theta_0)$$

which implies the triangle shown



$$\gamma^2 + \left(\frac{\dot{r}}{v_m}\right)^2 = \left(1-\frac{v_c^2}{v_m^2}\right)^2$$

$$\gamma^2 = \left(1-\frac{v_c^2}{v_m^2}\right)^2 - \left(\frac{\dot{r}}{v_m}\right)^2$$

$$\cos(\theta-\theta_0) = \gamma \Big/ \left(1-\frac{v_c^2}{v_m^2}\right)$$

$$\boxed{\begin{array}{l} \text{if } v_c = v_m \text{ we have } \dot{r} = 0 \\ \text{or } \quad r = \text{constant} = r_0 \end{array}}$$

That is $\boxed{v_c = v_m \rightleftarrows \text{orbit is a circle}}$

and as $r^2\dot{\theta} = r_0 v_m$

$$\boxed{\dot{\theta} = v_m / r_0}$$

The 'period' is given by the time to go 2π radians

$$T = \frac{2\pi}{\dot{\theta}} = \frac{2\pi r_0}{v_m} = \frac{2\pi r_0}{v_c}$$

but as $v_c = v_m$ for the circular orbit

$$T = \frac{2\pi r_0}{v_c} \quad , \quad v_c = \sqrt{\frac{K}{r_0}}$$

$$\therefore \quad T^2 = \frac{4\pi^2 r_0^3}{K}$$

$$T^2 = \left(\frac{4\pi^2}{K}\right) r_0^3$$

$$\boxed{\left(\frac{T^2}{r_0^3}\right) = \text{Constant} = \frac{4\pi^2}{K}}$$

which is Kepler's third law.

Games and Orbits - the geometry of the ellipse!

We have found that the orbit equation has the form:

$$\frac{r}{r_0} = \frac{v_m^2/v_c^2}{1 + \left(\frac{v_m^2}{v_c^2} - 1\right)\cos(\theta - \theta_0)}$$

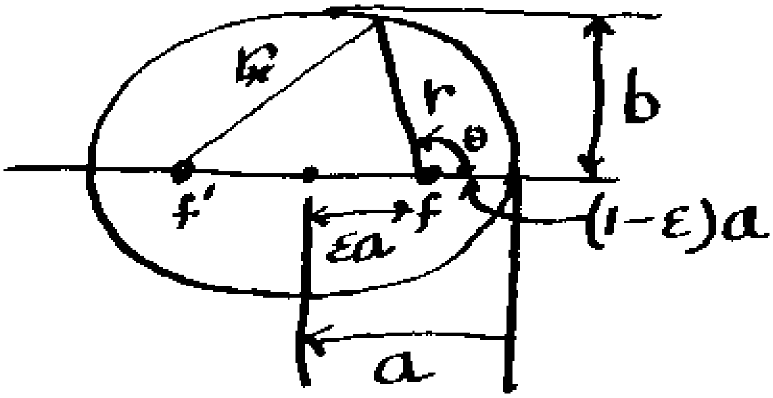
where $v_m = r_0 \dot{\theta}_0$ and $v_c = \sqrt{\frac{K}{r_0}}$

Note we can 'control' v_m but v_c is a natural constant - though even here we may be able to control r_0 which we generally take as the ~~point of~~ closest approach distance.

following Acheson we take $\theta_0 = 0$ and
$$e_* = \left(\frac{v_m}{v_c}\right)^2 - 1$$

Which Acheson calls the eccentricity authors do not always agree on ~~w~~ it is e_* or $\sqrt{e_*^2}$ which gets this name!

We now consider the following geometry



By definition an ellipse is defined such that given two 'focal' points f' and f. the ellipse is the curve such that $r_* + r$ the sum of the distance to the focal points is a constant! We can evaluate this constant when $\theta = 0$ so that:

$$r_* + r = \underbrace{(1-\varepsilon)a}_{r(0)} + \underbrace{2\varepsilon a + (1-\varepsilon)a}_{r_*(0)}$$

$$= 2a$$

By the law of cosines

$$r_*^2 = r^2 + (2\varepsilon a)^2 - 4\varepsilon a r \cos(\pi - \theta)$$

$$r_*^2 = r^2 + 4\varepsilon^2 a^2 + 4\varepsilon a r \cos\theta$$

now

$$r_* = 2a - r \quad \text{hence}$$

$$(2a-r)^2 = r^2 + 4\varepsilon^2 a^2 + 4\varepsilon a r\cos\theta$$

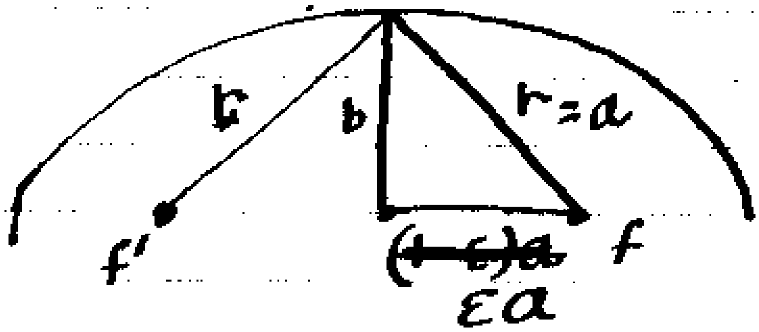
$$4a^2 - 4ar + \cancel{r^2} = \cancel{r^2} + 4\varepsilon^2 a^2 + 4\varepsilon a r\cos\theta$$

$$\cancel{4}a^2(1-\varepsilon^2) = \cancel{4}ar + \cancel{4}\varepsilon a r\cos\theta$$

$$r = \frac{a^2(1-\varepsilon^2)}{a(1+\varepsilon\cos\theta)}$$

$$\boxed{r = \frac{(1-\varepsilon^2)a}{1+\varepsilon\cos\theta}} \quad \text{Ellipse}$$

To find b, the semi minor distance consider the isoceles triangle when $r_* = r = s$



$$\therefore \quad 2s = 2a \quad \to s = a$$

$$\cancel{b^2 = a^2 - (1-\varepsilon)^2 a^2 = (1-1+2\varepsilon-\varepsilon^2)a^2}$$

$$\cancel{= \varepsilon(2-\varepsilon)a^2}$$

$$\cancel{b = \sqrt{\varepsilon(2-\varepsilon)}\, a}$$

$$\therefore \quad b^2 = a^2 - \varepsilon^2 a^2 = (1-\varepsilon^2)a^2$$

$$\boxed{b = \sqrt{(1-\varepsilon^2)}\, a}$$

also $\left(\frac{b}{a}\right)^2 = 1 - \varepsilon^2$

or $\cancel{\frac{b^2}{a^2}}$ $\boxed{\varepsilon^2 = 1 - \left(\frac{b}{a}\right)^2}$

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hence we may identify e_* with ε

$$r = \frac{(1-\varepsilon^2)a}{1+\varepsilon\cos\theta} = \frac{r_0 v_m^2/v_c^2}{1+\left(\frac{v_m^2}{v_c^2}-1\right)\cos\theta}$$

$$\therefore \quad \boxed{\varepsilon = \frac{v_m^2}{v_c^2} - 1}$$

$$(1-\varepsilon^2)a\cancel{\%} = r_0 v_m^2/v_c^2$$

or $\boxed{a = \cancel{\frac{1-\frac{r_0 v_m^2}{v_c^2}}{\left(\frac{v_m^2}{v_c^2}-1\right)^2}}} = \frac{r_0}{2-\frac{v_m^2}{v_c^2}}$

$v_c = \sqrt{K/r_0}$, v_m = Speed at closest approach $= \dot{\theta}_m r_0$

r_0 = distance at closest approach.