Notes on simple oscillatory systems

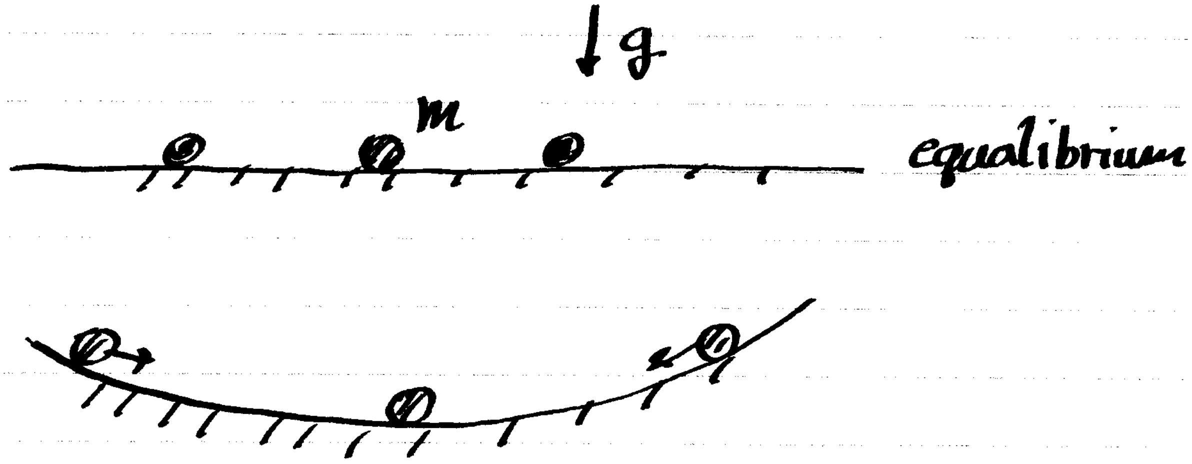
The basic cause of oscillation:

(a) inertia: carries a system over its equalibrium position

(b) restoring force: pushes a system toward its equalibrium position

(c) equalibrium - a state which is time independent - at least in some suitable reference frame.

The basic picture:



for some displacement we have (at least approximatively)

$$m\ddot{x} = f(x)$$

and if $f(x_0)=0$ so x_0 is an equalibrium position

Let $\xi = x - x_0$: the displacement from equalibrium

$f(x) = f(x_0 + \xi)$ $\quad \xi$ small so

$$m\ddot{x} = m\ddot{\xi} = \cancelto{0}{f(x_0)} + f'(x_0)\,\xi + \cdots$$

$$m\ddot{\xi} = f'(x_0)\,\xi$$

If f is a restoring force we expect $f'(x_0) < 0$.

By dimensional reasoning

$$\left[f'(x_0)\right] = \frac{\text{Force}}{\text{length}} = \frac{ML}{T^2}\,\frac{1}{L} = \frac{M}{T^2}$$

hence we get a frequency from

$$\text{freq} \sim 2\pi\sqrt{\frac{f'(x_0)}{m}}$$

or in general if $f = f(x, \lambda)$ where λ represents other variables (temprature)

$$\text{freq} \sim 2\pi\sqrt{\frac{1}{m}\frac{\partial f(x_0, \lambda_0)}{\partial x}}$$

We can also apply energy conservation if f is independent of t and $\dot{x}$

$$m\frac{dv}{dt} = f(x) \rightarrow d(mv) = f\,dt$$

$$\frac{dx}{dt} = v \longrightarrow d(x) = v\,dt$$

$$\therefore \quad \frac{d(mv)}{d(x)} = \frac{f}{v} \rightarrow v\frac{d}{dx}mv = f$$

$$d\left(\tfrac{1}{2}mv^2\right) = f\,dx$$

Let the potential energy be V

so $\quad dV = -f(x)\,dx$

$$\therefore \quad d\left(\tfrac{1}{2}mv^2 + V\right) = 0$$

$$\tfrac{1}{2}mv^2 + V = E. = E_0 \text{ a constant.}$$

where E_0 is the <u>energy</u> and $\frac{1}{2}mv^2$ is the <u>kinetic energy</u>. So.

$$v = \frac{dx}{dt} = \pm\left(\frac{2(E_0 - V(x))}{m}\right)^{1/2}$$

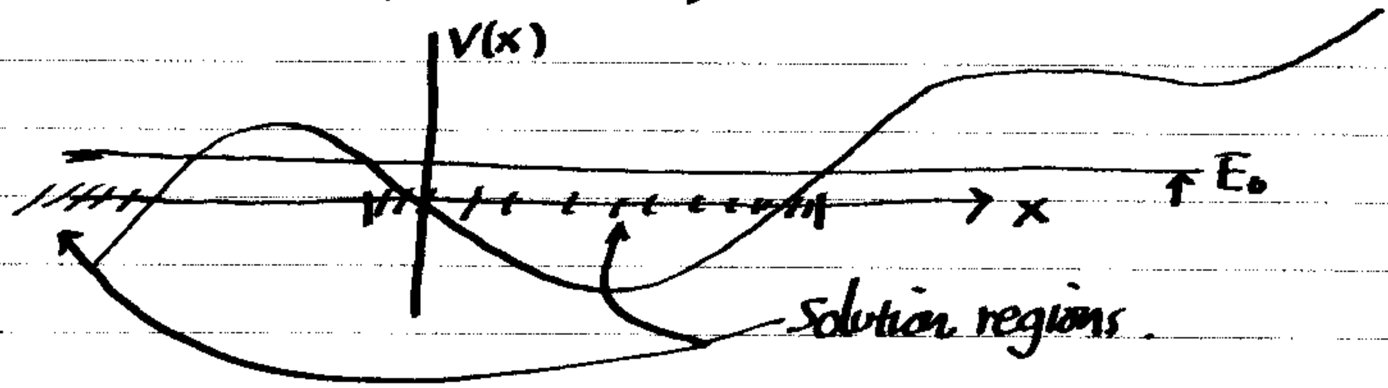
and hence $\quad dt = \pm\displaystyle\int\frac{dx}{\sqrt{\frac{2}{m}(E_0 - V(x))}}$

for a solution to exist we must have

$$V(x) \leq E_0$$

$\therefore$ the solution is in the set

$$\{ x \mid V(x) \leq E_0 \}$$

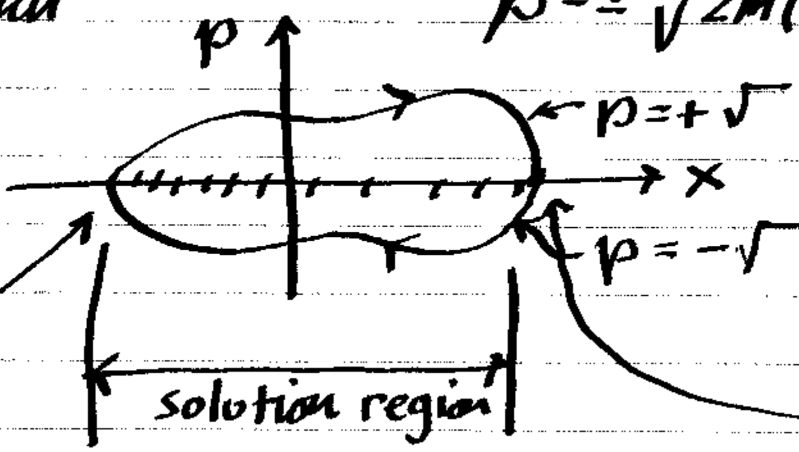


Sometimes it is useful to use the momentum

$$p = mv$$

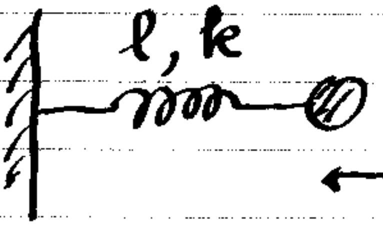
so that

$$p = \pm \sqrt{2m(E_0 - V(x))}$$



$V(\bar{x}) = E_0$, $p = 0$, equation for turning point

Example the <u>simple harmonic oscillator</u> SHO



$$\leftarrow -k(x - l) = f$$

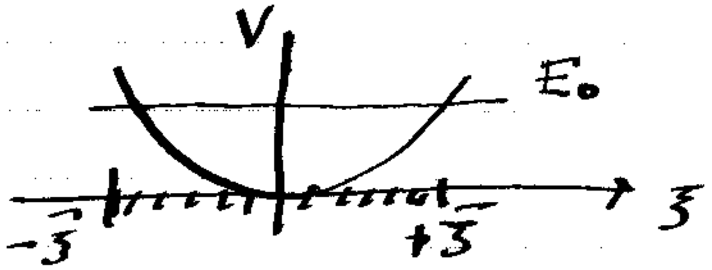
take $\xi = x - l$, $x_0 = l \to f(\xi) = -k\xi$
and $v = \dot{\xi}$

$$V = -\int_0^{\xi} f(\xi')\,d\xi' = k\int_0^{\xi} \xi'\,d\xi' = \frac{1}{2}k\xi^2$$

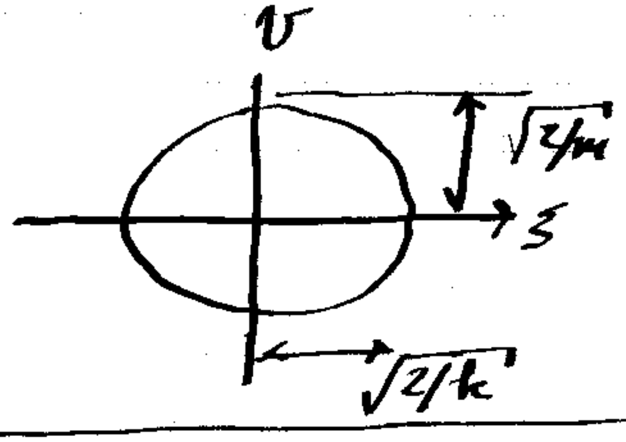
$$E_0 = \frac{1}{2}mv^2 + \frac{1}{2}k\xi^2 = \frac{1}{2m}p^2 + \frac{1}{2}k\xi^2$$

$$p = mv = m\dot{\xi}$$

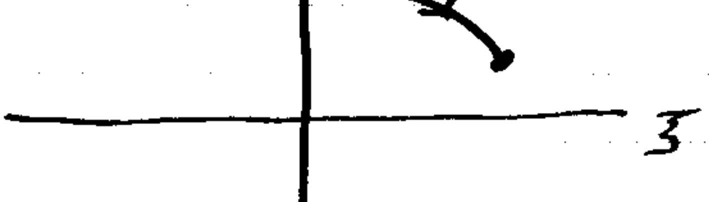
$$p = \pm\sqrt{2m\left(E_0 - \frac{1}{2}k\xi^2\right)}$$



$$E_0 = \frac{v^2}{2/m} + \frac{\xi^2}{2/k}$$



$\bar{\xi} = \pm\sqrt{\frac{2E_0}{k}}$ turning points

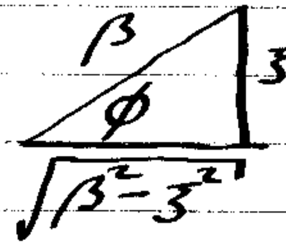


$$dt = \frac{d\xi}{\sqrt{\frac{2}{m}\left(E_0 - \frac{1}{2}k\xi^2\right)}}$$

$$\int_0^t dt' = \sqrt{\frac{m}{2}} \int_0^{\xi} \frac{d\xi'}{\sqrt{E_0 - \frac{1}{2}k\xi'^2}}$$

let $\beta^2 = 2E_0/k$ $\quad k/m = \omega_0^2$

$$\omega_0 t = \int_0^{\xi} \frac{d\xi'}{\sqrt{\beta^2 - \xi'^2}}$$



$$\left.\begin{aligned} \cos\phi &= \frac{1}{\beta}\sqrt{\beta^2 - \xi^2} \\ \sin\phi &= \xi/\beta \end{aligned}\right\} \qquad d\xi = \beta\cos\phi\, d\phi$$

hence $$\int_0^{\xi} \frac{d\xi'}{\sqrt{\beta^2 - \xi'^2}} = \int_{\sin^{-1}(0)}^{\sin^{-1}(\frac{\xi}{\beta})} \frac{\cancel{\beta\cos\phi}\, d\phi}{\cancel{\beta\cos\phi}}$$

$$= \int_0^{\sin^{-1}(\frac{\xi}{\beta})} d\phi = \sin^{-1}(\xi/\beta) = \omega t$$

hence $\xi = \beta \sin(\omega_0 t)$ or

$$\xi = \sqrt{\frac{2E_0}{k}} \sin\omega_0 t$$

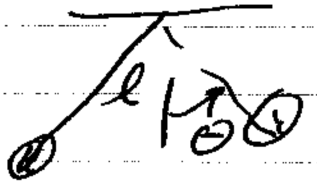
at $t=0$ take $v = v_0$, $\xi(0) = 0$

$$\xi = \sqrt{\frac{2}{k}\frac{1}{2}m v_0^2} \sin\omega_0 t \qquad \text{now use } \omega_0^2 = \frac{k}{m}$$

$$\boxed{\xi = \omega_0 v_0 \sin\omega_0 t}$$

<u>For the pendulum we have.</u>

$$m\ddot{\theta} + mg\sin\theta = 0$$



$$dV = -(-mg\sin\theta)\,d\theta$$

$$V = -mg\cos\theta + V(\theta=0)$$

if $V(0)=0 \rightarrow V(0)=mg$.

$$V = mg(1-\cos\theta)\ , \text{ let } \dot{\theta} = \Omega$$

$$\therefore \quad \frac{1}{2}m\Omega^2 + mg(1-\cos\theta) = E_0$$

We choose to use units where $m=1, g=1, l=1$

which implies as $[g] = [L/T^2]$,

Time Unit $\sqrt{\frac{g}{l}}$, mass unit m,

length unit l

So to get velocity multiply by

$$\frac{l}{\sqrt{\frac{g}{l}}} = \sqrt{\frac{l^3}{g}}\ , \quad \text{accel.} \rightarrow \frac{l}{\frac{g}{l}} = \frac{l^2}{g}.$$

So in these units

$$E_0 = \frac{1}{2}\Omega^2 + 1 - \cos\theta$$

as $[E] = \left[\frac{ML^2}{T^2}\right]$ the multiplier is

$$\frac{m\ell^2}{g/\ell} = \left(\frac{m\ell^3}{g}\right)$$ to get ~~proper~~ normal units.

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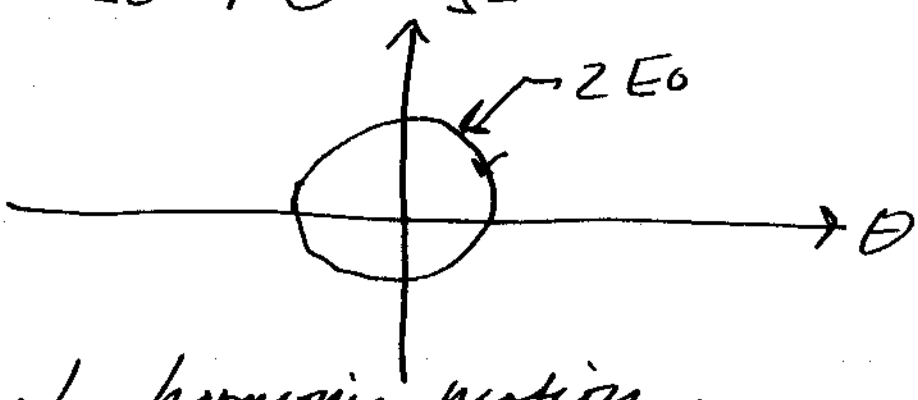
plots in (θ, Ω) phase space.

if θ small, E_0 small

$$E_0 = \frac{1}{2}\Omega^2 + 1 - \left(1 - \frac{1}{2}\theta^2 + \cdots\right)$$

$$= \frac{1}{2}\Omega^2 + \frac{1}{2}\theta^2$$

$$2E_0 = \Omega^2 + \theta^2$$



Simple harmonic motion.

Simple harmonic Oscillator in Complex form (Maybe)

$$\ddot{x} + \omega x = 0$$
$$\dot{x} = v$$
$$\dot{v} = -\omega x \qquad \text{take } \omega = 1$$
$$\cancel{z = \omega x + i}$$
$$\dot{x} = v$$
$$\dot{v} = -x$$
$$z = x + iv$$
$$\dot{x} + i\dot{v} = v - ix = -i(iv + x)$$
$$\dot{z} = -iz \quad \text{or} \quad \boxed{\dot{z} + iz = 0}$$
$$z = Ce^{it}$$
$$z(0) = (x_0 + iv_0) = C$$
$$\therefore \quad z = (x_0 + iv_0)e^{it}$$
$$z = (x_0 + iv_0)(\cos t + i\sin t)$$
$$= (x_0\cos t - v_0\sin t) + i(v_0\cos t + x_0\sin t)$$
$$\text{or} \qquad |z| = \sqrt{x_0^2 + v_0^2}$$
$$\arg(z_0) = \tan^{-1}\left(\frac{v_0}{x_0}\right)$$
$$z = \sqrt{x_0^2 + v_0^2}\,\exp\left\{i\left(t + \tan^{-1}\frac{v_0}{x_0}\right)\right\}$$

Dissapative System

$$E = \frac{1}{2} m v^2 + V$$

$$\frac{dE}{dt} = m v \dot{v} + \frac{dV}{dx} v$$

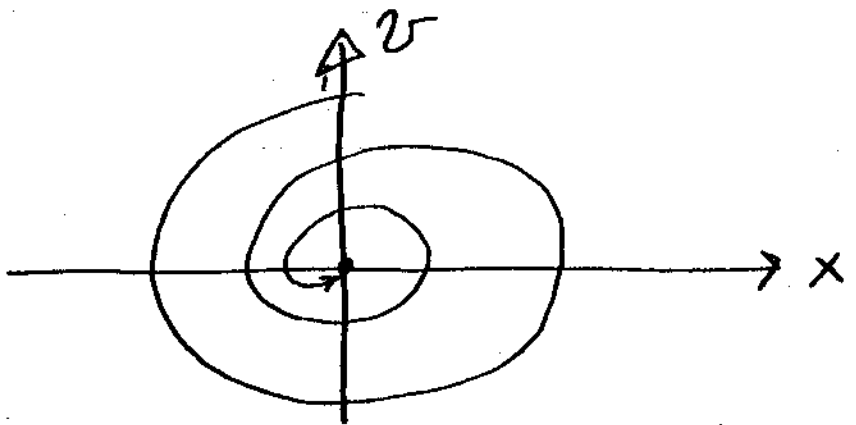
$$m\dot{v} = -\frac{dV}{dx} - D(x, \dot{x})$$

$$\frac{dE}{dt} = v\left(-\frac{dV}{dx} - D\right) + \frac{dV}{dx} v$$

$$\therefore \quad \boxed{\frac{dE}{dt} = -D(x,v)\, v}$$

if $D = \nu v$

$$\rightarrow \quad \frac{dE}{dt} = -\nu v^2 \leq 0 \quad \text{if } \nu > 0$$



So the system approaches a state of rest.

Forced motion

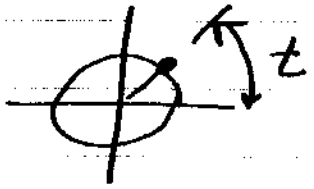
~~the~~ $\ddot{x} + \omega x = f(t)$ take $\omega = 1$

$$\dot{v} = -x + f$$
$$\dot{x} = v$$

$$i\dot{v} + \dot{x} = -ix + if + v$$
$$= -i(x + iv - f)$$

$$\boxed{\dot{z} + iz = if(t)}$$

complimentary solution e^{it}



let $z_p = w(t)e^{it}$

$$\dot{z}_p = \dot{w}e^{it} + wie^{it}$$

$$\dot{z}_p + iz_p = \dot{w}e^{it} + wie^{it} + i(iwe^{it}) = \dot{w}e^{it}$$
$$= if(t)$$

$$\therefore \dot{w} = +if(t)e^{-it}$$

$$\frac{dw}{dt} = i\, f(t) e^{-it}$$

$$w = i \int f(t) e^{-it}\, dt$$

$$\therefore \quad z_p = i e^{it} \int_0^t f(t') e^{-it'}\, dt'$$

$$= i \int_0^t f(t') e^{i(t-t')}\, dt'$$

$$z = C e^{it} + i \int_0^t f(t') e^{i(t-t')}\, dt'$$

$$\boxed{z(t) = (x_0 + i v_0) e^{it} + i \int_0^t f(t') e^{i(t-t')}\, dt'}$$

Suppose $f(t) = A \sin \beta t$

$$z_p = i e^{it} \int_0^t A \sin \beta t')\, e^{-it'}\, dt'$$

$$= A i e^{it} \int_0^t \left\{ \frac{1}{2i} (e^{i\beta t'} - e^{-i\beta t'}) e^{-it'}\, dt'\right.$$

$$= i \frac{Ae^{it}}{\cancel{2i}} \left\{ \frac{\int_0^t (e^{i(\beta-1)t'} - e^{-i(\beta+1)t'})}{2i} \right\} dt'$$

$$\cancel{= iAe^{it} \int}$$

$$Z_p = \frac{Ae^{it}}{2}\left\{\int_0^t e^{i(\beta-1)t'}dt' - \int_0^t e^{-i(\beta+1)t'}dt'\right\}$$

$$Z_p = \frac{Ae^{it}}{2}\left\{\frac{e^{i(\beta-1)t}-1}{i(\beta-1)} + \frac{e^{-i(\beta+1)t}-1}{i(\beta+1)}\right\}$$

$$= -\frac{iAe^{it}}{2}\left\{\frac{e^{i(\beta-1)t}}{\beta-1} + \frac{e^{-i(\beta+1)t}}{\beta+1} - \frac{1}{\beta-1} - \frac{1}{\beta+1}\right\}$$

$$= -\frac{iAe^{it}}{2}\left\{\frac{e^{i(\beta-1)t}}{\beta-1} + \frac{e^{-i(\beta+1)t}}{\beta+1} - \frac{2\beta}{\beta^2-1}\right\}$$

$$Z_p = \frac{-iA}{2}\left\{\frac{e^{i\beta t}}{\beta-1} + \frac{e^{-i\beta t}}{\beta+1} - \frac{2\beta}{\beta^2-1}\right\}$$

$$= -\frac{iA}{2}\left\{\frac{(\beta+1)e^{i\beta t}+(\beta-1)e^{-i\beta t}}{\beta^2-1} - \frac{2\beta}{\beta^2-1}\right\}$$

$$Z_p = -\frac{iA}{2(\beta^2-1)}\left\{(\beta+1)e^{i\beta t}+(\beta-1)e^{-i\beta t}-2\beta\right\}$$

$$Z_p = -\frac{iA}{2(\beta^2-1)}\left\{\beta(e^{i\beta t}+e^{-i\beta t}) + e^{i\beta t} - e^{-i\beta t} - 2\beta\right\}$$

$$Z_p = -\frac{iA}{\cancel{2}(\beta^2-1)}\left\{2\beta\cos\beta t + 2i\sin\beta t - 2\beta\right\}$$

$$Z_p = -\frac{iA\,2}{\cancel{2}(\beta^2-1)}\left\{\beta(\cos\beta t - 1) + i\sin\beta t\right\}$$

$$\therefore\ Z = (x_0 + iv_0)e^{it} - i\left(\frac{2A}{\cancel{2}(\beta^2-1)}\right)\left\{\beta(\cos\beta t - 1) + i\sin\beta t\right\}$$

$\cancel{Z = (x}$

$$z = (x_0 + iv_0)e^{it} + \frac{2A}{\cancel{2}(\beta^2-1)}\left\{\sin\beta t - i\beta(\cos\beta t - 1)\right\}$$

let $\beta = 1+\varepsilon$

$$(\beta-1)(\beta+1) = (1+\varepsilon-1)(1+\varepsilon+1) \to \varepsilon(2+\varepsilon)$$

$$\cos\beta t \to \cos(1+\varepsilon)t \to \cos t \cos\varepsilon t - \sin t \sin\varepsilon t$$

$$\to \cos t\left(1-\tfrac{1}{2}\varepsilon^2 t^2\right) - \sin t\,(\varepsilon t) + \cdots$$

<u>etc</u>

$$Z_p = -\frac{2Ai}{2(\beta^2-1)}\left\{\beta(\cos\beta t-1) + i\sin\beta t\right\}$$

<u>not immediatly valid if $\beta = 1$ (resonance.)</u>

$$f(t) = A\sin t$$

$$Z_p = i\int_0^t A\sin t' e^{i(t-t')}dt$$

$$= \frac{iAe^{it}}{2i}\int_0^t (e^{it'}-e^{-it'})e^{-it'}dt$$

$$Z_p = \frac{Ae^{it}}{2}\int_0^t \left\{1-e^{-2it'}\right\}dt$$

$$Z_p = \frac{1}{2}Ae^{it}\left[t' + \frac{1}{2i}e^{-2it'}\right]_0^t$$

$$= \frac{1}{2}At'e^{it} + \frac{1}{2}Ae^{it}\left(\frac{e^{-2it}-1}{2i}\right)$$

$$= \frac{1}{2}At'e^{it} + \frac{1}{2}A\frac{(e^{-it}-e^{it})}{2i}$$

$$= \frac{1}{2}At'e^{it} + \frac{1}{2}A\sin t$$

linear growth term. (resonance)