# Advanced Dynamics of Complex Systems with SOPHIA '03

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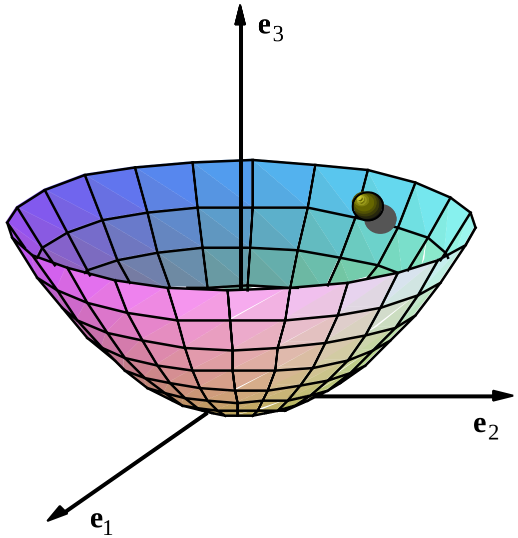
# Lecture 1: Maple




**Introductory example 1:** A particle is moving frictionless on a given surface, in this case a two-dimensional parabola. Analyse the motion and the constraint force.

**Dynamic law**

Newton's equation of motion:

$$\mathbf{R} - m\,\mathbf{a} = \mathbf{0}, \tag{1}$$

where

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}, \tag{2}$$

$$\mathbf{R} = \mathbf{R}_a + \mathbf{R}_c \quad \text{(applied+constraining forces)}, \tag{3}$$

$$\mathbf{R}_a = -mg\,\mathbf{e}_3,$$

$$\mathbf{R}_c = \quad \text{unknown??} \qquad \text{(not completely)}$$

## Constraints

The constraining surface:

$$\phi(\mathbf{r}, t) = x_3 - k(x_1^2 + x_2^2) = 0\,, \qquad k = \text{ const.}. \qquad (4)$$

where

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3\,.$$

## Generalised coordinates

Not all three coordinates are independent, only two are needed. It is also possible to choose two parameters $q_1$ and $q_2$ (or generalised coordinates). For example:

$$\mathbf{r} = \mathbf{r}(q_1, q_2, t) \rightarrow \begin{cases} x_1 = q_1 \cos q_2 \\ x_2 = q_1 \sin q_2\,. \\ x_3 = kq_1^2 \end{cases} \qquad (5)$$

## Tangent vectors

As for polar coordinates in a flat plane, these $q_1$ and $q_2$ also correspond to local directions in space, tangents to the coordinate lines on the surface.

$$\tau_1 = \frac{\partial \mathbf{r}}{\partial q_1} = \cos q_2 \mathbf{e}_1 + \sin q_2 \mathbf{e}_2 + 2kq_1 \mathbf{e}_3\,,$$

$$\tau_2 = \frac{\partial \mathbf{r}}{\partial q_2} = -q_1 \sin q_2 \mathbf{e}_1 + q_1 \cos q_2 \mathbf{e}_2\,. \qquad (6)$$

•What does 'local' mean?
•Is it necessary to normalise these tangent vectors?
•Are there more tangent vectors?

## Constraining force

The constraining force $\mathbf{R}_c$ keeps the particle on the surface. Its magnitude depends on the mass and the motion of the particle, but the direction is always orthogonal (or normal) to the surface.

A normal to the surface is given by:

$$\mathbf{N} = \tau_1 \times \tau_2$$

$$= -2kq_1^2 \cos q_2 \mathbf{e}_1 + -2kq_1^2 \sin q_2 \mathbf{e}_2 + q_1\left(\cos^2 q_2 + \sin^2 q_2\right)\mathbf{e}_3\,. \qquad (7)$$

Another way to find this direction is to use the gradient to the surface. Consider the surface (4) to be a level surface. The neighbouring surface of higher 'value' points in the direction

$$\phi = \frac{\partial \phi}{\partial x_1}\mathbf{e}_1 + \frac{\partial \phi}{\partial x_2}\mathbf{e}_2 + \frac{\partial \phi}{\partial x_3}\mathbf{e}_3$$

$$= -2kx_1\mathbf{e}_1 - 2kx_2\mathbf{e}_2 + \mathbf{e}_3\,. \qquad (8)$$

Comparing the vectors, we see that $\mathbf{N} = q_1 \;\; \phi$, they are indeed parallel. The constraining force can then be written as $\mathbf{R}_c = \lambda\mathbf{N}/|\mathbf{N}|$, where now only the 'magnitude' $\lambda$ is unknown.

## Strategy

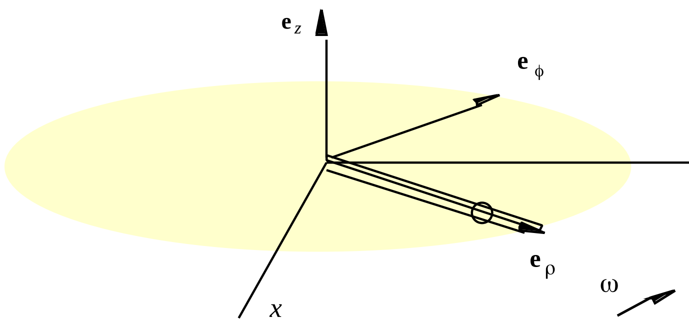
Since the constraining force $\mathbf{R}_c$ is partly unknown but orthogonal to the surface $\mathbf{R}_c \cdot \tau_k = 0$, take the scalar product of the dynamical equations with two independent tangent vectors $\tau_k$ (need not be exactly the ones we derived):

$$\left(\mathbf{R}_a - m\,\mathbf{a}\right) \cdot \tau_k = 0\,.$$

These are two equations in this case, with two generalised coordinates. When they are known the complete motion of the particle is known, hence from (1),

$$\lambda + \left(\mathbf{R}_a - m\mathbf{a}\right) \cdot \frac{\mathbf{N}}{|\mathbf{N}|} = 0$$

provides an equation for the magnitude $\lambda$ of the constaining force.




**Introductory example 2: The bead sliding along a rotating wire**
Consider the motion and forces in the plane $z=0$. Using a cylindrical basis, the acceleration and constraining force take the form:

$$\mathbf{a} = \left(\ddot{\rho} - \rho\omega^2\right)\mathbf{e}_\rho + \left(2\dot{\rho}\omega\right)\mathbf{e}_\phi, \qquad \mathbf{R}_c = \lambda\mathbf{e}_\phi.$$

Note that the gravitational force is orthogonal to the considered plane.
In this problem the bead is completely described by the coordinate $\rho$. The single tangent vector corresponding to this coordinate is $\tau_\rho = \mathbf{e}_\rho$.

Following the strategy, we first solve the tangent projection of Newton's equation:

$$m\left(\ddot{\rho} - \rho\omega^2\right) = 0 \qquad\qquad \rho(t) = c_1 e^{\omega t} + c_2 e^{-\omega t}.$$

Then the magnitude of the constraining force is found:

$$\lambda(t) = 2m\,\omega\dot{\rho} = 2m\,\omega^2\left(c_1 e^{\omega t} - c_2 e^{-\omega t}\right).$$

Observation
The **velocity** of the bead:

$$\mathbf{v} = \dot{\rho}\mathbf{e}_\rho + \rho\omega\mathbf{e}_\phi,$$

•is <u>not</u> orthogonal to the constraining force, whence resulting in an important energy time rate ('power'): $P = \mathbf{R}_c \cdot \mathbf{v} = \lambda(t)\rho\omega$.

• is <u>not</u> parallel to the tangent vector $\tau_\rho = \mathbf{e}_\rho$.

# MAPLE

Take a second look at the first problem of the introduction. Now we use plain Maple.

```
> restart:
  with(linalg):
```

```
Warning, the protected names norm and trace have been
redefined and unprotected
```

## The steps follow section 1.4.1: "The Parabola revisited."

## Cartesian coordinates parametrized

```
> x1:=q1*cos(q2);
x2:=q1*sin(q2);
x3:=k*q1^2;
```

$$x1 := q1\cos(q2)$$

$$x2 := q1\sin(q2)$$

$$x3 := k\,q1^2$$

## Parameters are time dependent

```
> toTimeFunction:=
{q1=q1(t),q1t=diff(q1(t),t),q1tt=diff(q1(t),t,t),q2
=q2(t),q2t=diff(q2(t),t),q2tt=diff(q2(t),t,t)}:
```

## Simplifying the resulting expressions

```
toTimeExpression:={q1(t)=q1,diff(q1(t),t)=q1t,diff(
q1(t),t,t)=q1tt,q2(t)=q2,diff(q2(t),t)=q2t,diff(q2(
t),t,t)=q2tt}:
```

## Velocity components

```
>
v1:=subs(toTimeExpression,diff(subs(toTimeFunction,
x1),t));
```

$$v1 := q1t\cos(q2) - q1\sin(q2)\,q2t$$

```
>
v2:=subs(toTimeExpression,diff(subs(toTimeFunction,
x2),t));
v3:=subs(toTimeExpression,diff(subs(toTimeFunction,
x3),t));
```

$$v2 := q1t\sin(q2) + q1\cos(q2)\,q2t$$

$$v3 := 2\,k\,q1\,q1t$$

\>

## Acceleration components

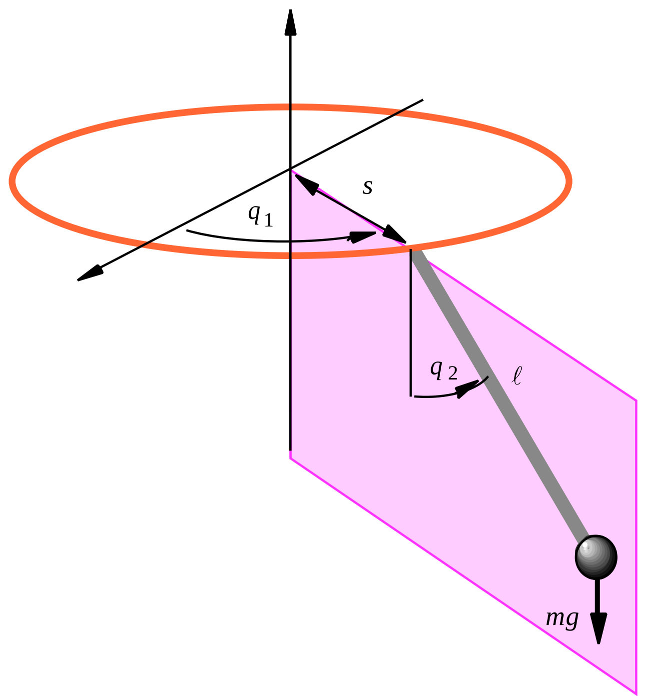
```
>for j from 1 to 3 do
a||j:=subs(toTimeExpression,diff(subs(toTimeFunctio
n,v||j),t)) od:

>for j from 1 to 3 do a||j od;
```

$$q1tt\cos(q2) - 2\,q1t\sin(q2)\,q2t - q1\cos(q2)\,q2t^2 - q1\sin(q2)\,q2tt$$

$$q1tt\sin(q2) + 2\,q1t\cos(q2)\,q2t - q1\sin(q2)\,q2t^2 + q1\cos(q2)\,q2tt$$

$$2\,k\,q1t^2 + 2\,k\,q1\,q1tt$$

**New example: Pendulum hanging from a rotating disc**
This is a similar problem. However we show how to use a simple form of vectors with Maple.




## The steps follow section 1.4.2: Using Lists. Pendulum on circular support:

**Combine components and form vectors using MAPLE lists.**

**Position vector:**

```
>x:=(s+l*sin(q2))*cos(q1):
y:=(s+l*sin(q2))*sin(q1):
z:=-l*cos(q2):
>r:=[x,y,z]:
```

### Velocity and acceleration

```
>rs:=subs(toTimeFunction,r):
vs:=map(diff,rs,t):
v:=subs(toTimeExpression,vs):
as:=map(diff,vs,t):
a:=subs(toTimeExpression,as):
```

### Show components

```
>a[1];
```

$$-l\sin(q2)\,q2t^2\cos(q1)+l\cos(q2)\,q2tt\cos(q1)-2\,l\cos(q2)\,q2t\sin(q1)\,q1t$$
$$-(s+l\sin(q2))\cos(q1)\,q1t^2-(s+l\sin(q2))\sin(q1)\,q1tt$$

**Two ways to form pt(=m*a), the rate of change of the momentum vector.**
**First way:**

```
>pt:=evalm(m*a):
>pt[3];
```

$$m\,(l\cos(q2)\,q2t^2+l\sin(q2)\,q2tt)$$

**Second way:**

```
>Pt:=map(x->m*x,a):
>Pt[3];
```

$$m\,(l\cos(q2)\,q2t^2+l\sin(q2)\,q2tt)$$

### Tangent vectors

```
>tau1:=map(diff,r,q1);
```

$$\tau1 := [-(s+l\sin(q2))\sin(q1),\,(s+l\sin(q2))\cos(q1),\,0]$$

```
>tau2:=map(diff,r,q2);
```

$$\tau2 := [l\cos(q2)\cos(q1),\,l\cos(q2)\sin(q1),\,l\sin(q2)]$$

### Projection of 'inertial force' onto the tangent plane

```
>Pt1:=multiply(pt,tau1);
Pt1:=simplify(multiply(pt,tau1));
Pt2:=simplify(multiply(pt,tau2)):
```

$$Pt1 := -m\,(-l\sin(q2)\,q2t^2\cos(q1)+l\cos(q2)\,q2tt\cos(q1)$$
$$-2\,l\cos(q2)\,q2t\sin(q1)\,q1t-(s+l\sin(q2))\cos(q1)\,q1t^2$$
$$-(s+l\sin(q2))\sin(q1)\,q1tt)\,(s+l\sin(q2))\sin(q1)+m\,($$
$$-l\sin(q2)\,q2t^2\sin(q1)+l\cos(q2)\,q2tt\sin(q1)$$
$$+2\,l\cos(q2)\,q2t\cos(q1)\,q1t-(s+l\sin(q2))\sin(q1)\,q1t^2$$
$$+(s+l\sin(q2))\cos(q1)\,q1tt)\,(s+l\sin(q2))\cos(q1)$$

$$Pt1 := -m\,(-2\,l\cos(q2)\,q2t\,q1t\,s-2\,l^2\cos(q2)\,q2t\,q1t\sin(q2)-q1tt\,s^2$$
$$-2\,q1tt\,s\,l\sin(q2)-q1tt\,l^2+q1tt\,l^2\cos(q2)^2)$$

**Projection of the gravitational force–the only applied force. The gravitational force is:**

```
>Rg:=[0, 0, -m*g]:
```

```
>R1:=simplify(multiply(Rg,tau1));
R2:=simplify(multiply(Rg,tau2));
```

$$R1 := 0$$

$$R2 := -m\,g\,l\sin(q2)$$

### Equations

```
>Eq1:=Pt1=R1:Eq2:=Pt2=R2:
```

**These equations contain second derivatives. Standard numerical routines solving differential equations use a set of first-order coupled equations. We introduce generalized speeds u1=q1t and u2=q2t, to eliminate higher derivatives. First we isolate the second-order derivatives in our equations:**

```
>Eqs:=solve({Eq1,Eq2},{q1tt,q2tt}):
```

**We arrive at a system of first-order equations in standard form.**

```
>
Eqs:=simplify(subs({q1tt=u1t,q2tt=u2t,q1t=u1,q2t=u2
},Eqs)):
>Eqs[1];
```

$$u1t = 2\,\frac{u1\,u2\,l\,(2\cos(q2)\,s + l\sin(2\,q2))}{-2\,s^2 - 4\,s\,l\sin(q2) - l^2 + l^2\cos(2\,q2)}$$

```
>Eqs[2];
```

$$u2t = \frac{1}{2}\,\frac{2\cos(q2)\,u1^2\,s + l\,u1^2\sin(2\,q2) - 2\,g\sin(q2)}{l}$$

**4 first-order differential equations for MAPLE.**

```
>state:=Eqs union {q1t=u1,q2t=u2};
```

$$state := \left\{ q1t = u1,\ q2t = u2,\ u2t = \frac{\cos(q2)\,u1^2\,s + l\cos(q2)\,u1^2\sin(q2) - g\sin(q2)}{l},\right.$$
$$\left. u1t = -2\,\frac{(s - l\sin(q2))\,u1\,u2\cos(q2)\,l}{s^2 - l^2 + l^2\cos(q2)^2} \right\}$$

**Extend**

```
>toTimeFunction:=toTimeFunction union
{u1=u1(t),u1t=diff(u1(t),t),u2=u2(t),u2t=diff(u2(t)
,t)}:
```

**Choose parameter values and initial conditions**

```
> param:={s=1,l=4,g=10}:
>statep:=subs(param,state):
>statepEq:=subs(toTimeFunction,statep);
```

$$statepEq := \left\{\frac{\partial}{\partial t}\,q1(t) = u1(t),\ \frac{\partial}{\partial t}\,q2(t) = u2(t),\right.$$
$$\frac{\partial}{\partial t}\,u2(t) = \frac{1}{4}\cos(q2(t))\,u1(t)^2 + \frac{1}{2}\,u1(t)^2\sin(2\,q2(t)) - \frac{5}{2}\sin(q2(t)),$$
$$\left.\frac{\partial}{\partial t}\,u1(t) = -8\,\frac{\cos(q2(t))\,u2(t)\,u1(t)}{1 + 4\sin(q2(t))}\right\}$$

```
>initc:= {q1(0)=0,q2(0)=1,u1(0)=0.5,u2(0)=0}:
```

**Put into one set for MAPLE**

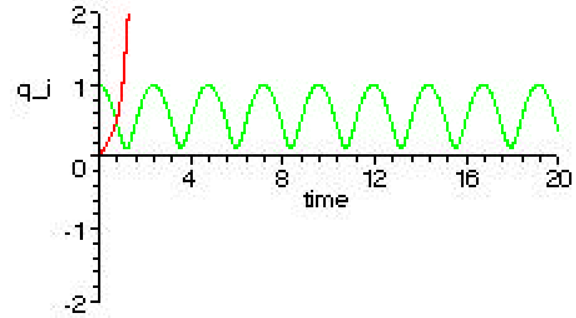
```
> deqns:=statepEq union initc:
```

**Solving and plotting**

```
>
st:=dsolve(deqns,{q1(t),q2(t),u1(t),u2(t)},type=num
eric,output=procedurelist);
```

$st :=$ **proc**$(rkf45\_x)$ ... **end**

```
>with(plots,odeplot);

>odeplot(st,[[t,q1(t)],[t,q2(t)]],0..20,view=[0..20
,-2...2],numpoints=100,labels=[time,q_i]);
```



# Lecture 2: **Configurations**

## Orthonormal basis

The space can be organised using 3 orthonormal basis vectors. They are composed into a reference triad

$$n = \begin{pmatrix}\mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3\end{pmatrix}. \qquad (1.1)$$

Its transpose composition is:

$$n^T = \begin{pmatrix}\mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3\end{pmatrix}.$$

Here are some <u>algebraic rules</u> with these triads:

### Scalar product operations

- $$n \cdot n^T = \begin{pmatrix}\mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3\end{pmatrix} \cdot \begin{pmatrix}\mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3\end{pmatrix} = \mathbf{n}_1 \cdot \mathbf{n}_1 + \mathbf{n}_2 \cdot \mathbf{n}_2 + \mathbf{n}_3 \cdot \mathbf{n}_3 = 3.$$

- $$n^T \cdot n = \begin{pmatrix}\mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3\end{pmatrix} \cdot \begin{pmatrix}\mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3\end{pmatrix} = \begin{pmatrix}\mathbf{n}_1 \cdot \mathbf{n}_1 & \mathbf{n}_1 \cdot \mathbf{n}_2 & \mathbf{n}_1 \cdot \mathbf{n}_3 \\ \mathbf{n}_2 \cdot \mathbf{n}_1 & \mathbf{n}_2 \cdot \mathbf{n}_2 & \mathbf{n}_2 \cdot \mathbf{n}_3 \\ \mathbf{n}_3 \cdot \mathbf{n}_1 & \mathbf{n}_3 \cdot \mathbf{n}_2 & \mathbf{n}_3 \cdot \mathbf{n}_3\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix} \qquad (1.2)$$

### Vector product operations

- $$n \times n^T = \begin{pmatrix}\mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3\end{pmatrix} \times \begin{pmatrix}\mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3\end{pmatrix} = \mathbf{0}.$$

- $$n^T \times n = \begin{pmatrix}\mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3\end{pmatrix} \times \begin{pmatrix}\mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3\end{pmatrix} = \begin{pmatrix}\mathbf{n}_1 \times \mathbf{n}_1 & \mathbf{n}_1 \times \mathbf{n}_2 & \mathbf{n}_1 \times \mathbf{n}_3 \\ \mathbf{n}_2 \times \mathbf{n}_1 & \mathbf{n}_2 \times \mathbf{n}_2 & \mathbf{n}_2 \times \mathbf{n}_3 \\ \mathbf{n}_3 \times \mathbf{n}_1 & \mathbf{n}_3 \times \mathbf{n}_2 & \mathbf{n}_3 \times \mathbf{n}_3\end{pmatrix} = \begin{pmatrix}\mathbf{0} & \mathbf{n}_3 & -\mathbf{n}_2 \\ -\mathbf{n}_3 & \mathbf{0} & \mathbf{n}_1 \\ \mathbf{n}_2 & -\mathbf{n}_1 & \mathbf{0}\end{pmatrix}.$$

Note the order of the triads!
The 'matrix order' does not always mean that matrix rules are applicable, for example:

$$\left[n^T \times n\right]^T = -n^T \times n,$$

and the matrix rule for vector products is inapplicable.

### Dyadic operation

Here is another <u>algebraic rule</u> with these triads:

- $$n n^T = \begin{pmatrix}\mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3\end{pmatrix} \begin{pmatrix}\mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3\end{pmatrix} = \mathbf{n}_1\mathbf{n}_1 + \mathbf{n}_2\mathbf{n}_2 + \mathbf{n}_3\mathbf{n}_3 = \mathbf{U}$$

In the last operation we obtain a new quantity, the <u>unit dyad</u> $\mathbf{U}$.

Let the reference vectors further satisfy the 'right-hand rule':

$$(\mathbf{n}_1 \times \mathbf{n}_2) \cdot \mathbf{n}_3 = 1. \qquad (1.3)$$

## Basis expansion

An arbitrary vector $\mathbf{w}$ may be decomposed along the reference vectors $\mathbf{n}_i$ :

$$\mathbf{w}={}^n w_1\mathbf{n}_1+{}^n w_2\mathbf{n}_2+{}^n w_3\mathbf{n}_3$$

$$=\begin{pmatrix}\mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3\end{pmatrix}\begin{pmatrix}{}^n w_1\\ {}^n w_2\\ {}^n w_3\end{pmatrix}=n\,{}^n w \qquad (1.4)$$

$$=\begin{pmatrix}{}^n w_1 & {}^n w_2 & {}^n w_3\end{pmatrix}\begin{pmatrix}\mathbf{n}_1\\ \mathbf{n}_2\\ \mathbf{n}_3\end{pmatrix}={}^n w^T n^T. \qquad (1.4')$$

The components of a vector are organised in columns ${}^n w$ :

$${}^n w=\begin{pmatrix}{}^n w_1\\ {}^n w_2\\ {}^n w_3\end{pmatrix}. \qquad (1.5)$$

${}^n w$ is only a particular representation of the vector $\mathbf{w}$ in terms of the reference basis vectors, not the complete vector itself.

## Components

The operation to obtain ${}^n w$ from the original vector is:

$${}^n w=n^T\cdot\mathbf{w}. \qquad (1.6)$$

and for obtaining a single component ${}^n w_i$ :

$${}^n w_i=\mathbf{w}\cdot\mathbf{n}_i=\mathbf{n}_i\cdot\mathbf{w}.$$

## Norm

Finally the norm (or length) of a vector is defined as usual

$$|\mathbf{w}|=\sqrt{\mathbf{w}\cdot\mathbf{w}},$$

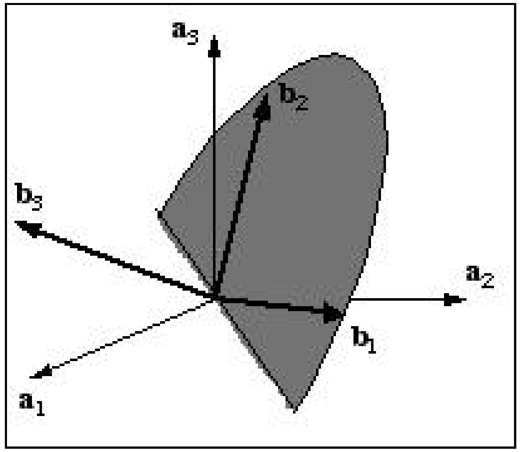
if the vector is real valued.

Using the triad decompositions (1.4) and (1.4') we find in general

$$|\mathbf{w}|=\sqrt{{}^n w^T n^T\cdot n\,{}^n w}=\sqrt{{}^n w^T\,{}^n w},$$

where for a unit base triad $n^T\cdot n$ is a unit matrix.

# Transformation of basis




Let $a$ and $b$ be two alternative reference triads, composed of mutually orthogonal unit basis vectors. Any vector $\mathbf{w}$ can be decomposed in either set of basis vectors, from which we obtain the alternative representaions ${}^a w$ and ${}^b w$.

We now see how these are related to the relative orientations in space of the corresponding reference triads $a$ and $b$. Each basis vector in $b$ can be expanded in $\mathbf{a}_i$.

We have

$$\mathbf{b}_1=(\mathbf{a}_1\cdot\mathbf{b}_1)\mathbf{a}_1+(\mathbf{a}_2\cdot\mathbf{b}_1)\mathbf{a}_2+(\mathbf{a}_3\cdot\mathbf{b}_1)\mathbf{a}_3$$

$$=\begin{pmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{pmatrix}\begin{pmatrix}\mathbf{a}_1\cdot\mathbf{b}_1\\ \mathbf{a}_2\cdot\mathbf{b}_1\\ \mathbf{a}_3\cdot\mathbf{b}_1\end{pmatrix}$$

$$= a\begin{pmatrix}\mathbf{a}_1\cdot\mathbf{b}_1\\ \mathbf{a}_2\cdot\mathbf{b}_1\\ \mathbf{a}_3\cdot\mathbf{b}_1\end{pmatrix} = aa^T\cdot\mathbf{b}_1,$$

$$\mathbf{b}_2 = \begin{pmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{pmatrix}\begin{pmatrix}\mathbf{a}_1\cdot\mathbf{b}_2\\ \mathbf{a}_2\cdot\mathbf{b}_2\\ \mathbf{a}_3\cdot\mathbf{b}_2\end{pmatrix} = a\begin{pmatrix}\mathbf{a}_1\cdot\mathbf{b}_2\\ \mathbf{a}_2\cdot\mathbf{b}_2\\ \mathbf{a}_3\cdot\mathbf{b}_2\end{pmatrix} = aa^T\cdot\mathbf{b}_2,$$

$$\mathbf{b}_3 = a\begin{pmatrix}\mathbf{a}_1\cdot\mathbf{b}_3\\ \mathbf{a}_2\cdot\mathbf{b}_3\\ \mathbf{a}_3\cdot\mathbf{b}_3\end{pmatrix} = aa^T\cdot\mathbf{b}_3.$$

Note the operation of a unit dyad $aa^T$ represented in the basis $a$. Together these transformation equations can be written:

$$b = \begin{pmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3\end{pmatrix} = a\begin{pmatrix}\mathbf{a}_1\cdot\mathbf{b}_1 & \mathbf{a}_1\cdot\mathbf{b}_2 & \mathbf{a}_1\cdot\mathbf{b}_3\\ \mathbf{a}_2\cdot\mathbf{b}_1 & \mathbf{a}_2\cdot\mathbf{b}_2 & \mathbf{a}_2\cdot\mathbf{b}_3\\ \mathbf{a}_3\cdot\mathbf{b}_1 & \mathbf{a}_3\cdot\mathbf{b}_2 & \mathbf{a}_3\cdot\mathbf{b}_3\end{pmatrix} = aa^T\cdot b.$$

Formally we introduce the direction-cosine matrix $R_{ab}$:

$$b = a(a^T\cdot b) = aR_{ab}. \qquad (1.7\text{-}8)$$

Each column of $R_{ab}$ is an $a$-representation of a unit base vector, which is orthogonal to the ones of the other columns.

•The sum of squared components of any column is unity.
•The scalar product of two different columns vanishes.

Such a matrix is called an **orthogonal matrix**.

The inverse transformation matrix $R_{ba}$ is obtained explicitly, in analogy to the above steps, by interchanging $a$ :s and $b$ :s. With the short notation:

$$a = b(b^T\cdot a) = bR_{ba}.$$

We see the inverse property of $R_{ba}$ using also (1.7-8):

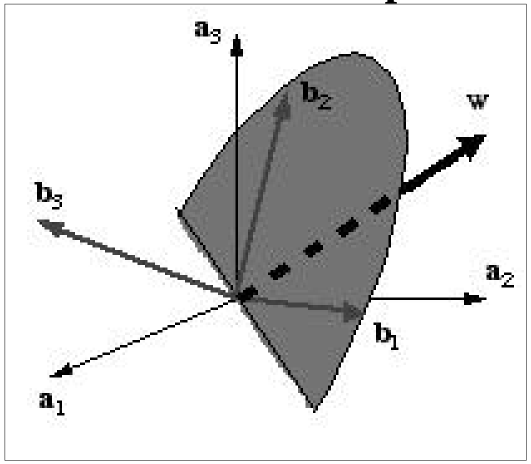
$$a = bR_{ba} = aR_{ab}R_{ba} = a.$$

The inverse matrix is obtaned by transposing the original one:

$$R_{ba} = b^T\cdot a = \left(a^T\cdot b\right)^T = R_{ab}^T.$$

Thus: $\quad R_{ab} = R_{ba}^T = R_{ba}^{-1}. \qquad (1.9)$

# Transformation of components




When we know how to transform the basis vectors we can convert between representaions ${}^a w$ and ${}^b w$ of a vector $\mathbf{w}$. From (1.6) and (1.4) we get

$${}^b w = b^T\cdot\mathbf{w} = b^T\cdot\left(a\,{}^a w\right) = \left(b^T\cdot a\right){}^a w,$$

hence

${}^b w = R_{ba}\,{}^a w$, [remember $b = a(a^T\cdot b) = aR_{ab}$ in(1.7-8) ].

The reverse transformation involves $R_{ab}$:

$${}^a w = R_{ab}\,{}^b w. \qquad (1.10)$$

# Solved Problems

## P1.1

Given a reference triad $a$ construct a triad $b$ such that $\mathbf{b}_1$ is parallel to $\mathbf{a}_1 + \mathbf{a}_2$ and $\mathbf{b}_2$ is anti-parallel to $\mathbf{a}_1 - \mathbf{a}_2$. Find the matrix which transforms reference from $b$ to $a$ with $\mathbf{b}_3 = \mathbf{a}_3$.

### Solution

Clearly the constructed unit vectors in $b$ are,

$$\mathbf{b}_1 = \frac{1}{\sqrt{2}}\left(\mathbf{a}_1 + \mathbf{a}_2\right),\ \mathbf{b}_2 = -\frac{1}{\sqrt{2}}\left(\mathbf{a}_1 - \mathbf{a}_2\right),\ \text{and } \mathbf{b}_3 = \mathbf{a}_3 .$$

It follows by inserting into the definition of $R_{ab}$ that

$$R_{ab} = a^T \cdot b = \begin{pmatrix} \mathbf{a}_1\cdot\mathbf{b}_1 & \mathbf{a}_1\cdot\mathbf{b}_2 & \mathbf{a}_1\cdot\mathbf{b}_3 \\ \mathbf{a}_2\cdot\mathbf{b}_1 & \mathbf{a}_2\cdot\mathbf{b}_2 & \mathbf{a}_2\cdot\mathbf{b}_3 \\ \mathbf{a}_3\cdot\mathbf{b}_1 & \mathbf{a}_3\cdot\mathbf{b}_2 & \mathbf{a}_3\cdot\mathbf{b}_3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

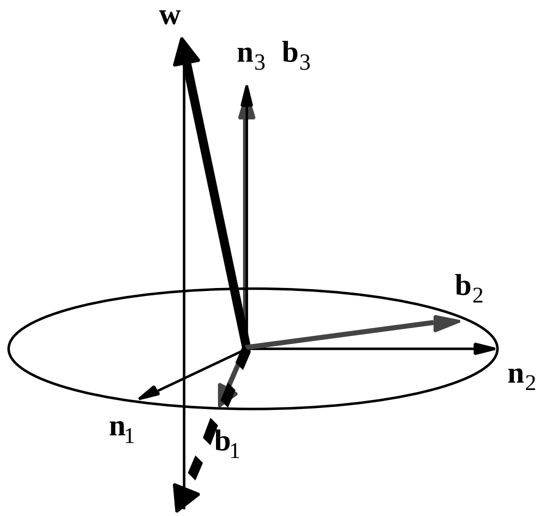
But $R_{ba}$ transforms the triads from $b$ to $a$. Transposition gives:

$$\boxed{R_{ba} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}}.$$

## P1.2

Derive the transformation matrix between Cartesian and cylindrical (polar) reference triads with a common $z$-axis. Apply the transformation to an arbitrary position vector.

### Solution




Let the Cartesian coordinate system be spanned by the triad $n$, so that the cylindrical basis vectors in the triad $b$ are rotated an amount $\varphi$ about the common $\mathbf{n}_3$-direction. Then

$$R_{nb} = n^T \cdot b = \begin{pmatrix} \mathbf{n}_1\cdot\mathbf{b}_1 & \mathbf{n}_1\cdot\mathbf{b}_2 & \mathbf{n}_1\cdot\mathbf{b}_3 \\ \mathbf{n}_2\cdot\mathbf{b}_1 & \mathbf{n}_2\cdot\mathbf{b}_2 & \mathbf{n}_2\cdot\mathbf{b}_3 \\ \mathbf{n}_3\cdot\mathbf{b}_1 & \mathbf{n}_3\cdot\mathbf{b}_2 & \mathbf{n}_3\cdot\mathbf{b}_3 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

When this is applied to a position vector represented in $b$ as

$${}^b r = \begin{pmatrix} \rho \\ 0 \\ z \end{pmatrix},$$

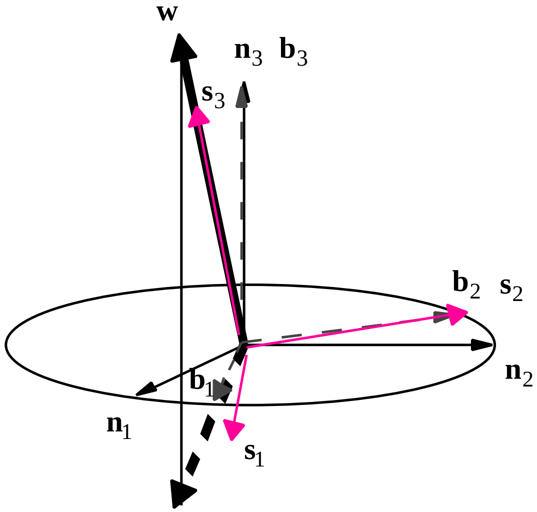
we get the Cartesian representation from ${}^n r = R_{nb}\,{}^b r$:

$${}^n r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \rho \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} \rho\cos\varphi \\ \rho\sin\varphi \\ z \end{pmatrix}.$$

## P1.3

Derive the transformation matrix between Cartesian and spherical basis triads. Apply it to an arbitrary position vector representation ${}^{s}r$ in the spherical triad.

## Solution




Consider three reference triads $n$, $b$ and $s$. Let $n$ be the Cartesian triad. We introduce $b$ as an auxiliary reference triad rotated relative to $n$ an amount $\varphi$ about the common $\mathbf{n}_3$ direction. Then the spherical triad $s$ is obtained rotated relative to $b$ an amount $\vartheta$ about the common $\mathbf{b}_2$ direction.

The two simple rotation matrices are:

$$R_{nb} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } \quad R_{bs} = \begin{pmatrix} \cos\vartheta & 0 & \sin\vartheta \\ 0 & 1 & 0 \\ -\sin\vartheta & 0 & \cos\vartheta \end{pmatrix}$$

They combine, according to the matrix product rule to the Cartesian-spherical basis transformation:

$$R_{ns} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\vartheta & 0 & \sin\vartheta \\ 0 & 1 & 0 \\ -\sin\vartheta & 0 & \cos\vartheta \end{pmatrix} = \begin{pmatrix} \cos\varphi\cos\vartheta & -\sin\varphi & \cos\varphi\sin\vartheta \\ \sin\varphi\cos\vartheta & \cos\varphi & \sin\varphi\sin\vartheta \\ -\sin\vartheta & 0 & \cos\vartheta \end{pmatrix}.$$

With the spherical triad the position vector is $\mathbf{r} = \rho\mathbf{s}_3$. When subsequently $R_{ns}$ is applied to ${}^{s}r$ we get ${}^{n}r$.:

$${}^{n}r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\varphi\cos\vartheta & -\sin\varphi & \cos\varphi\sin\vartheta \\ \sin\varphi\cos\vartheta & \cos\varphi & \sin\varphi\sin\vartheta \\ -\sin\vartheta & 0 & \cos\vartheta \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \rho \end{pmatrix} = \begin{pmatrix} \rho\cos\varphi\sin\vartheta \\ \rho\sin\varphi\sin\vartheta \\ \rho\cos\vartheta \end{pmatrix}.$$

Comparing the components in both triads, we see that the length $|\mathbf{r}| = \rho$ obviously is the same.

### ——P1.3 The spherical frame—with – SOPHIA ——

```
> restart;
> read sophia21_3_V5;
```

*sophia21_3 - 26 May 1998*

**Sophia definition of the sequence of simple rotations for the transformation**

```
> rotList:=[[N,B,3,phi],[B,S,2,theta]];
> chainSimpRot(rotList);
```

$rotList := [[N, B, 3, \varphi], [B, S, 2, \theta]]$

*Frame relation between N and B defined!*

*Frame relation between B and S defined!*

*true*

**Calling the matrix of transformation**

```
> Rmx(N,B);
```

$$\begin{bmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
> Rmx(B,S);
```

$$\begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

```
> Rmx(N,S);
> Rmx(S,N);
```

$$\begin{bmatrix} \cos(\varphi)\cos(\theta) & -\sin(\varphi) & \cos(\varphi)\sin(\theta) \\ \sin(\varphi)\cos(\theta) & \cos(\varphi) & \sin(\varphi)\sin(\theta) \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$\begin{bmatrix} \cos(\varphi)\cos(\theta) & \sin(\varphi)\cos(\theta) & -\sin(\theta) \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ \cos(\varphi)\sin(\theta) & \sin(\varphi)\sin(\theta) & \cos(\theta) \end{bmatrix}$$

```
> inverse(Rmx(N,S));
```

$$\begin{bmatrix} \frac{\cos(\varphi)\cos(\theta)}{\cos(\varphi)^2\cos(\theta)^2+\sin(\varphi)^2\cos(\theta)^2+\sin(\varphi)^2\sin(\theta)^2+\cos(\varphi)^2\sin(\theta)^2}, \frac{\sin(\varphi)\cos(\theta)}{\cos(\varphi)^2\cos(\theta)^2+\sin(\varphi)^2\cos(\theta)^2+\sin(\varphi)^2\sin(\theta)^2+\cos(\varphi)^2\sin(\theta)^2}, -\frac{\sin(\theta)}{\cos(\theta)^2+\sin(\theta)^2} \\ -\frac{\sin(\varphi)}{\sin(\varphi)^2+\cos(\varphi)^2}, \frac{\cos(\varphi)}{\sin(\varphi)^2+\cos(\varphi)^2}, 0 \\ \frac{\cos(\varphi)\sin(\theta)}{\cos(\varphi)^2\cos(\theta)^2+\sin(\varphi)^2\cos(\theta)^2+\sin(\varphi)^2\sin(\theta)^2+\cos(\varphi)^2\sin(\theta)^2}, \frac{\sin(\varphi)\sin(\theta)}{\cos(\varphi)^2\cos(\theta)^2+\sin(\varphi)^2\cos(\theta)^2+\sin(\varphi)^2\sin(\theta)^2+\cos(\varphi)^2\sin(\theta)^2}, \frac{\cos(\theta)}{\cos(\theta)^2+\sin(\theta)^2} \end{bmatrix}$$

```
> simplify(inverse(Rmx(N,S)));
```

$$\begin{bmatrix} \cos(\varphi)\cos(\theta) & \sin(\varphi)\cos(\theta) & -\sin(\theta) \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ \cos(\varphi)\sin(\theta) & \sin(\varphi)\sin(\theta) & \cos(\theta) \end{bmatrix}$$

# Evectors (Euclidian vectors in Sophia)

**In the spherical coordinate frame we have**

```
> rS:=S &ev [0,0,rho];
```

$$rS := [[0, 0, \rho], S]$$

**The same vector in the original (Newtonian) frame**

```
> rN:= N &to rS;
```

$$rN := [[\cos(\varphi)\sin(\theta)\,\rho, \sin(\varphi)\sin(\theta)\,\rho, \cos(\theta)\,\rho], N]$$

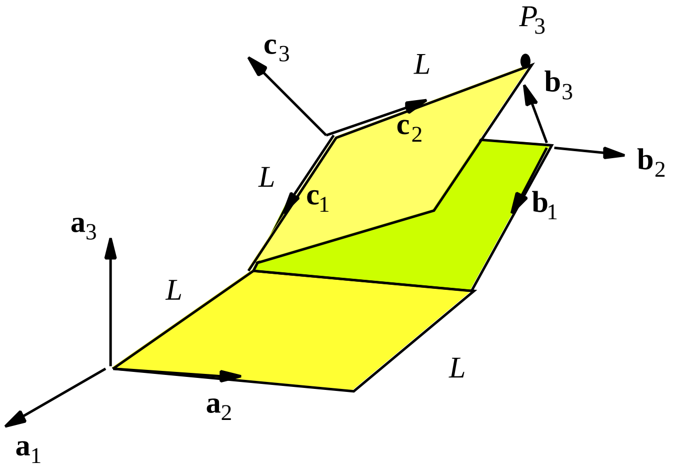
# Frames and configuration

A reference frame is a reference triad + a reference point (local origin). We need particular reference frames (inertial frames) to formulate Newton's dynamical laws. But we may use other convenient reference frames in intermediate steps, for example a body-fixed reference frame which exploit the symmetry of a body. Such reference frames may depend on time or some physical angle or some other parameters.

## Sophia in action: ML: Section 2.10.2.

**Example: Specifying positions**

**This is the 'Illustration' problem in ML, page 41-43.**
**Three square plates are connected. Express the geometrical displacement vector from the corner of the first plate at the A-origin to the most distant corner of the third plate. See figure.**




**Inspection of the 'simple' rotations involved gives:**

```
>rotList:=[[A,B,2,q1],[B,C,1,q2]];
chainSimpRot(rotList);
```

$$rotList := [[A, B, 2, q1], [B, C, 1, q2]]$$

*Frame relation between A and B defined!*

*Frame relation between B and C defined!*

*true*

**Define relevant Evectors:**

```
>r01:= A &ev [-L,0,0];
r12:= B &ev [-L,0,0];
r23:= C &ev [0,L,0];
r03:= (r01 &++ r12) &++ r23;
```

$$r01 := [[-L, 0, 0], A]$$

$$r12 := [[-L, 0, 0], B]$$

$$r23 := [[0, L, 0], C]$$

$$r03 := [[-\cos(q1)\,L - L, -\sin(q2)\sin(q1)\,L + L, -\cos(q2)\sin(q1)\,L], C]$$

**Express Evector in the A-frame:**

```
>A &to r03;
```

$$[[L\,(-\cos(q1) - 1 + \sin(q1)\sin(q2)), \cos(q2)\,L, L\,(\sin(q1) + \cos(q1)\sin(q2))], A]$$

-----------------------------------------------

# Lecture 3:

# Velocity, angular velocity and dyads

The problem is that observed rates of changes with respect to time or relevant parameters are measured to different values depending on which reference frame is used.

## Generalised coordinates

We denote derivatives with respect to a generalised coordinate $q$ relative to a reference frame $N$, by

$$ {}^{N}\frac{\partial}{\partial q} . \qquad (2.1)$$

Most important in our case is the total time derivative:

$${}^{N}\frac{\mathrm{d}}{\mathrm{d}t} \equiv {}^{N}\dot{\ } \; .$$

If $N$ is an inertial frame we may sometimes omit the frame superscript.

Derivatives of scalar quantities or matrices of scalars do not depend on reference frames. Hence we may omit the left superscript in this case as well. For example: a projection component of a specific vector on a specific axis is a scalar.

For a position or displacement vector $\mathbf{r}$, on the other hand, we find:

$${}^{B}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r} = {}^{B}\frac{\mathrm{d}}{\mathrm{d}t}\left(b\,{}^{b}r\right) = b\,\frac{\mathrm{d}\,{}^{b}r}{\mathrm{d}t}, \qquad (2.2)$$

whereas

$${}^{N}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r} = {}^{N}\frac{\mathrm{d}}{\mathrm{d}t}\left(b\,{}^{b}r\right) = \left({}^{N}\frac{\mathrm{d}b}{\mathrm{d}t}\right){}^{b}r + b\,\frac{\mathrm{d}\,{}^{b}r}{\mathrm{d}t}. \qquad (2.3')$$

We find an additional term here. Written in terms of the full vector $\mathbf{r}$, we have

$$\begin{aligned}{}^{N}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r} &= {}^{B}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r} + \left({}^{N}\frac{\mathrm{d}b}{\mathrm{d}t}\,b^{T}\right)\cdot\mathbf{r} \\ &= {}^{B}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r} + {}^{N}\Omega^{B}\cdot\,\mathbf{r}. \qquad (2.3)\end{aligned}$$

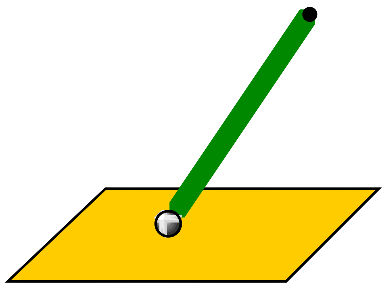
In the last equation a new symbol ${}^{N}\Omega^{B}$ for a dyadic quantity has been introduced. Similar relations are obtained if we consider differentiations with respect to generalised coordinates.

The operation by the dyad ${}^{N}\Omega^{B}\cdot$ on a vector can be understood as an operation by a related vector ${}^{N}\omega^{B}\times$ (the angular velocity vector) as we shall see later on.

# Solved Problems

## P2.1

Consider a thin rigid rod connected at one end to a spherical joint fixed at the origin of a reference frame $N$. Find the position and velocity vectors of a point on the other end relative to $N$. Use SOPHIA.



### Solution

We let the Cartesian coordinate system be spanned by the reference triad $n$ and we introduce two auxiliary reference triads:

- $b$ rotated relative to $n$ an amount $q_1$ about the common direction $\mathbf{n}_3$.
- $s$ rotated relative to $b$ an amount $q_2$ about the common direction $\mathbf{b}_2$.

Using SOPHIA we will get:

$$\mathbf{v} = \ell\dot{q}_1 \sin q_2 \mathbf{s}_2 + \ell\dot{q}_2\mathbf{s}_1,$$

with respect to frame $N$. This result is now shown using SOPHIA.

### Initiation

```
> read sophia21_3_V5;
```

sophia21_3 - 26 May 1998

### Frame relations

```
> dependsTime(q1,q2):
&rot[N,B,3,q1]:&rot[B,S,2,q2]:
```

### Position and velocity

```
> r:=S &ev [0,0,L];
```

$r := [[0, 0, L], S]$

**Direct calculation (using frame derivative)**

```
>v:=&simp (N &fdt r);
```

$v := [[q2t\,L, q1t\sin(q2)\,L, 0], S]$

**but also in this way**

### Angular velocity vectors for frame rotations

```
> wNS:= S &to (N &aV S);
> wNS:= B &to (N &aV S);
```

$wNS := [[-\sin(q2)\,q1t, q2t, q1t\cos(q2)], S]$

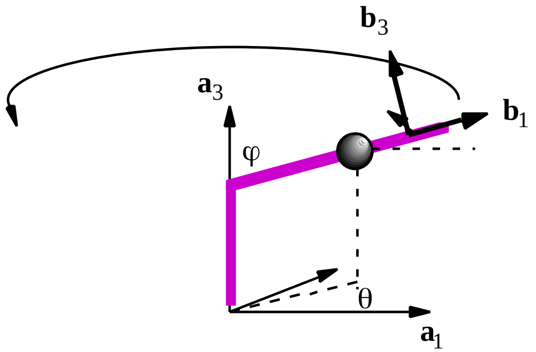
$wNS := [[0, q2t, q1t], B]$

**Velocity from angular velocity**

```
> v:=wNS &xx r;
```

$v := [[q2t\,L, q1t\sin(q2)\,L, 0], S]$

-----------------------------------------




## P2.2

A rigid pipe is bent an angle $\varphi$ at some point along its length. Let one end be fixed at the origin of a reference frame $A$. Assume that the pipe spins about an axis through the origin and the bend. Consider also a ball free to move along the slanted segment. Find the position and velocity vectors of the ball relative to $A$.

## Solution

Define convenient reference triads $a$ (fixed in $A$) and $b$ (fixed at the slanted segment). No common fixed axis in these triads, so we need an auxiliary one $f$, say. Let $\mathbf{f}_3 = \mathbf{a}_3$ and $\mathbf{f}_1$ traces the shadow on the 'floor'.

Then $\mathbf{b}_2 = \mathbf{f}_2$ and the rest of $b$ is in the figure. Note that $\mathbf{b}_1$ is tilted an angle $\vartheta = \pi/2 - \varphi$, which is time independent.

## ----------SOPHIA----------------

```
>restart;
> read sophia21_3_V5;
```

**Frames**

```
> &rot[A,F1,3,q1]:
> &rot[F1,B,2,Pi/2-phi]:
> dependsTime(q1,q2,u1,u2):
```

**Angular velocity of frames**

```
>  wAB:= A &to (A &aV B);
>  wAB:= B &to (A &aV B);
```

$wAB := [[0, 0, q1t], A]$

$wAB := [[-\cos(\varphi)\,q1t, 0, q1t\sin(\varphi)], B]$

**Position**

```
>  r:=B &ev [q2,0,0] &++ (A &ev [0,0,h]);
```

$r :=$

$[[\cos(q1)\sin(\varphi)\,q2, \sin(q1)\sin(\varphi)\,q2, -\cos(\varphi)\,q2 + h], A]$

**and velocity**?

```
> v:=&simp (A &fdt r);
> v:=&simp (wAB &xx r);
> v:=simplify (wAB &xx r);
```

$v := [[-\sin(q1)\,q1t\sin(\varphi)\,q2 + \cos(q1)\sin(\varphi)\,q2t,$
$\cos(q1)\,q1t\sin(\varphi)\,q2 + \sin(q1)\sin(\varphi)\,q2t, -\cos(\varphi)\,q2t], A]$

$v := [[-\sin(q1)\,q1t\sin(\varphi)\,q2, \cos(q1)\,q1t\sin(\varphi)\,q2, 0], A]$

$v := [[-\sin(q1)\,q1t\sin(\varphi)\,q2, \cos(q1)\,q1t\sin(\varphi)\,q2, 0], A]$

**Understanding the velocity now?**

```
> VA:=(B &ev [q2t,0,0]) &++ (wAB &xx r);
> VA:= A &fdt r;
```

$VA := [[\cos(q1)\sin(\varphi)\,q2t$
$- (\cos(\varphi)^2\,q1t + q1t\sin(\varphi)^2)\sin(q1)\sin(\varphi)\,q2,$
$\sin(q1)\sin(\varphi)\,q2t$
$+ (\cos(\varphi)^2\,q1t + q1t\sin(\varphi)^2)\cos(q1)\sin(\varphi)\,q2,$
$-\cos(\varphi)\,q2t], A]$

$VA := [[-\sin(q1)\, q1t \sin(\varphi)\, q2 + \cos(q1) \sin(\varphi)\, q2t,$
$\cos(q1)\, q1t \sin(\varphi)\, q2 + \sin(q1) \sin(\varphi)\, q2t, -\cos(\varphi)\, q2t], A]$

```
> VA :=B &to VA;
```

$VA := [[q2t, q1t \sin(\varphi)\, q2, 0], B]$

-----------------------------------------

# More on dyads

A dyad is simply a pair of vectors, written in a definite order $\mathbf{AB}$, $\mathbf{A}$ being the antecent vector and $\mathbf{B}$ the consequent one. The dot-product operation with another vector $\mathbf{v}$ can be performed in two ways and two simultaneous scalar operations with $\mathbf{v}$ and $\mathbf{w}$ likewise:

$$\mathbf{w}\cdot[\mathbf{AB}]\cdot\mathbf{v} = (\mathbf{w}\cdot\mathbf{A})(\mathbf{B}\cdot\mathbf{v}),$$
$$\mathbf{v}\cdot[\mathbf{AB}]\cdot\mathbf{w} = (\mathbf{v}\cdot\mathbf{A})(\mathbf{B}\cdot\mathbf{w}).$$

A dyadic is a linear combination of dyads. In fact, any dyad can be expressed as a dyadic in terms of basis vectors. Let

$$\mathbf{A} = A_1\mathbf{n}_1 + A_2\mathbf{n}_2 + A_3\mathbf{n}_3,$$
$$\mathbf{B} = B_1\mathbf{n}_1 + B_2\mathbf{n}_2 + B_3\mathbf{n}_3.$$

Then,

$$\begin{aligned}\mathbf{AB} &= A_1B_1\mathbf{n}_1\mathbf{n}_1 + A_1B_2\mathbf{n}_1\mathbf{n}_2 + A_1B_3\mathbf{n}_1\mathbf{n}_3\\ &+ A_2B_1\mathbf{n}_2\mathbf{n}_1 + A_2B_2\mathbf{n}_2\mathbf{n}_2 + A_2B_3\mathbf{n}_1\mathbf{n}_3\\ &+ A_3B_1\mathbf{n}_3\mathbf{n}_1 + A_3B_2\mathbf{n}_3\mathbf{n}_2 + A_3B_3\mathbf{n}_3\mathbf{n}_3\ .\end{aligned}$$

In any vector expansion the expansion coefficients are a kind of scalar products. So also for the dyadic expansions.

To make our formalism work we write $\mathbf{A} = n\ {}^nA$ and $\mathbf{B}={}^nB^T\ n^T$, so that

$$\mathbf{AB} = n\,({}^nA\,{}^nB^T)\,n^T\ . \quad \text{(please check!)}$$

The quantity inside the bracket is now a matrix.

**Generalisation**

For any matrix:

$${}^nD = \begin{pmatrix} {}^nD_{11} & {}^nD_{12} & {}^nD_{13}\\ {}^nD_{21} & {}^nD_{22} & {}^nD_{23}\\ {}^nD_{31} & {}^nD_{32} & {}^nD_{33}\end{pmatrix}, \qquad (2.8)$$

we define a dyad:

$$\begin{aligned}\mathbf{D} &= n\,{}^nDn^T = {}^nD_{11}\mathbf{n}_1\mathbf{n}_1 + {}^nD_{12}\mathbf{n}_1\mathbf{n}_2 + {}^nD_{13}\mathbf{n}_1\mathbf{n}_3 \ldots\\ &+ {}^nD_{31}\mathbf{n}_3\mathbf{n}_1 + {}^nD_{32}\mathbf{n}_3\mathbf{n}_2 + {}^nD_{33}\mathbf{n}_3\mathbf{n}_3\ . \qquad (2.9)\end{aligned}$$

We may here also define the transpose of a dyad according to:

$$\mathbf{D}^T = n\,{}^nD^Tn^T. \qquad (2.10)$$

Examples:

a) If ${}^nD$ is a unit matrix, then $\mathbf{D} = \mathbf{U}$, the **unit dyad.**

b) If $\mathbf{D} = \mathbf{n}_1\mathbf{n}_1 + \mathbf{n}_2\mathbf{n}_2$, its action on an arbitrary vector $\mathbf{w}$ is

$$\mathbf{D}\cdot\mathbf{w} = \mathbf{w}\cdot\mathbf{D} = w_1\mathbf{n}_1 + w_2\mathbf{n}_2\,.$$

This is a **vector projection** of the vector on to the $\mathbf{n}_1,\mathbf{n}_2$-plane. $\mathbf{D}$ is a **projection dyad**.

**Further results**

An important thing to remember is that a physical vector or a physical dyad are quantities which are independent of reference triads. But their representations are not. What happens to the different representations of a dyad? Let $a$ and $b$ be two alternative base triads. Then

$$\mathbf{D} = a\,{}^aDa^T = b\,{}^bDb^T.$$

Consequently,

$${}^aD = a^T\cdot b\,{}^bDb^T\cdot a = R_{ab}\,{}^bDR_{ba}\,, \qquad (2.11)$$

from the definitions (1.7-9) of the direction cosine matrices.

Note that a dyad can apply to a single vector, thereby producing a new vector. It follows from equations (1.4') and (2.10) that

$$\mathbf{D}\cdot\mathbf{w} = \mathbf{w}\cdot\mathbf{D}^T \qquad (2.12)$$

# Lecture 4:
## Velocity calculations—antisymmetric dyads

If the bodyfixed triad $b$ is moving with time, the Newtonian velocity calculation performes according to

$$^N\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r} = {}^B\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r} + \left({}^N\frac{\mathrm{d}b}{\mathrm{d}t}b^T\right)\cdot\mathbf{r}.$$

We obtain a dyad operating on the original position vector.

The new fundamental dyad ${}^N\Omega^B = \left({}^N\frac{\mathrm{d}b}{\mathrm{d}t}b^T\right) = \dot{b}b^T$ (the last member of the definition is a simplified notation) is anti-symmetric. To see this, we first note that

$$\left({}^N\Omega^B\right)^T = \left(\dot{b}b^T\right)^T = b\dot{b}^T.$$

Furthermore, the unit dyad expanded in the $b$ -triad

$$\mathbf{U} = bb^T,$$

is time independent and , hence, the relation

$$\mathbf{0} = \dot{b}b^T + b\dot{b}^T.$$

is always true.

The first term is ${}^N\Omega^B$ and the last term is our $\left({}^N\Omega^B\right)^T$ and the equation shows the antisymmetry

$$\left({}^N\Omega^B\right)^T = -{}^N\Omega^B.$$

### P2.3

Assume that $\mathbf{D}$ is a real antisymmetric dyad, i.e. $\mathbf{D}^T = -\mathbf{D}$ or $\mathbf{D}\cdot\mathbf{w} = -\mathbf{w}\cdot\mathbf{D}$ if $\mathbf{w}$ is a vector. Show that the eigenvalues of $\mathbf{D}$ are $0, \pm i\vartheta$ for some real quantity $\vartheta$ .

### Solution

**Matrix formulation of the eigenvalue problem**

Let $\mathbf{w} = n\,{}^n w$ and $\mathbf{D} = n\,{}^n D n^T$.

The dyad eigenvalue problem

$$n\,{}^n D n^T \cdot n\,{}^n w = \left(n\,{}^n w\right)\lambda$$

reduces to

$$n\,{}^n D\,{}^n w = \left(n\,{}^n w\right)\lambda\ .$$

Finally we eliminate the $n$ basis to find

$$^n D\ {}^n w = {}^n w\,\lambda\ ,$$

which is a matrix formulation involving the matrix ${}^n D$ and the column ${}^n w$ .

The matrix representation of an arbitrary dyad in a reference triad $n$ is given by

$$^n D = \begin{pmatrix} 0 & {}^n D_{12} & {}^n D_{13} \\ -{}^n D_{12} & 0 & {}^n D_{23} \\ -{}^n D_{13} & -{}^n D_{23} & 0 \end{pmatrix}.$$

The eigenvalue equation is

$$\begin{vmatrix} -\lambda & {}^n D_{12} & {}^n D_{13} \\ -{}^n D_{12} & -\lambda & {}^n D_{23} \\ -{}^n D_{13} & -{}^n D_{23} & -\lambda \end{vmatrix} = 0\,,$$

or in expanded form:

$$\begin{aligned} 0 &= (-\lambda)\left(\lambda^2 + {}^n D_{23}^2\right) - {}^n D_{12}\left(\lambda\,{}^n D_{12} + {}^n D_{23}\,{}^n D_{13}\right) \\ &\qquad + {}^n D_{13}\left({}^n D_{12}\,{}^n D_{23} - \lambda\,{}^n D_{13}\right) \\ &= (-\lambda)\left(\lambda^2 + {}^n D_{23}^2\right) - {}^n D_{12}\lambda\,{}^n D_{12} - {}^n D_{13}\lambda\,{}^n D_{13} \\ &= (-\lambda)\left(\lambda^2 + {}^n D_{23}^2 + {}^n D_{12}^2 + {}^n D_{13}^2\right). \end{aligned}$$

Hence the eigenvalues of $\mathbf{D}$ are:

$$0,\quad \pm i\vartheta\left(= \pm i\sqrt{{}^n D_{23}^2 + {}^n D_{12}^2 + {}^n D_{13}^2}\right).$$

## P2.4

Assume that $\mathbf{D}$ is a real antisymmetric dyad with eigenvalues $0, \pm i\vartheta$. Show that an *invariant expression* for $\mathbf{D}$ is $(\vartheta\mathbf{e}_3 \times \mathbf{U})$, where $\mathbf{U}$ is a unit dyad and $\mathbf{e}_3$ is the unit eigen vector for the zero eigenvalue.

## Solution

Let

$$ {}^nD = \begin{pmatrix} 0 & {}^nD_{12} & {}^nD_{13} \\ -{}^nD_{12} & 0 & {}^nD_{23} \\ -{}^nD_{13} & -{}^nD_{23} & 0 \end{pmatrix}. $$

We determine the normalised eigenvector ${}^ae_3 = (X\ Y\ Z)^T$ corresponding to the vanishing eigenvalue. The equations are:

$$ \begin{pmatrix} 0 & {}^nD_{12} & {}^nD_{13} \\ -{}^nD_{12} & 0 & {}^nD_{23} \\ -{}^nD_{13} & -{}^nD_{23} & 0 \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. $$

The two first rows are written as:

$$ {}^nD_{12}Y + {}^nD_{13}Z = 0, $$

$$ -{}^nD_{12}X + {}^nD_{23}Z = 0. $$

Let arbitrarily $Z = -{}^nD_{12}$, then the two equations result in $Y = {}^nD_{13}$ and $X = -{}^nD_{23}$. The real normalisation factor is simply $1/\vartheta$, according to the previous problem, which then gives the normalised vector:

$$ {}^ne_3 = \frac{1}{\vartheta}\left(-{}^nD_{23} \quad {}^nD_{13} \quad -{}^nD_{12}\right)^T = \frac{1}{\vartheta}\left({}^nD_{32} \quad {}^nD_{13} \quad {}^nD_{21}\right)^T $$

Construction of the full vector gives $\mathbf{e}_3 = n\,{}^ne_3$.

### The invariant form

There is a real basis $(\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3)$ in which the antisymmetric matrix with eigenvalues $0, \pm i\vartheta$ takes the similar form:

$$ {}^eD = \begin{pmatrix} 0 & -\vartheta & 0 \\ \vartheta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. $$

The corresponding dyad is:

$$ \mathbf{D} = e\,{}^eDe^T = \vartheta(\mathbf{e}_2\mathbf{e}_1 - \mathbf{e}_1\mathbf{e}_2) $$

$$ = (\vartheta\mathbf{e}_3 \times \mathbf{U}), $$

where

$$ \mathbf{e}_3 \times \mathbf{U} = \mathbf{e}_3 \times (\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3) = \mathbf{e}_2\mathbf{e}_1 - \mathbf{e}_1\mathbf{e}_2 + 0. $$

In the end we can write the dyadic operation on a vector as a vector cross product:

$$ \boxed{\mathbf{D}\cdot\mathbf{w} = (\vartheta\mathbf{e}_3 \times \mathbf{U})\cdot\mathbf{w} = \vartheta_{\mathbf{D}} \times \mathbf{w}} $$

with $\vartheta_{\mathbf{D}} = \vartheta\mathbf{e}_3$, i.e. in the original basis

$$ \vartheta_{\mathbf{D}} = \vartheta\mathbf{e}_3 = \vartheta n\,{}^ne_3 = n\left({}^nD_{32} \quad {}^nD_{13} \quad {}^nD_{21}\right)^T. $$

## P2.5

Find an explicit formula for the vector corresponding to the antisymmetric dyad ${}^N\Omega^B$.

## Solution

By definition

$$ {}^N\Omega^B = \dot{b}b^T. $$

Its matrix representation in the $b$ basis is:

$$ b^T\cdot{}^N\Omega^B\cdot b = b^T\cdot\dot{b} = $$

$$ = \begin{pmatrix} 0 & \mathbf{b}_1\cdot\dot{\mathbf{b}}_2 & \mathbf{b}_1\cdot\dot{\mathbf{b}}_3 \\ \mathbf{b}_2\cdot\dot{\mathbf{b}}_1 & 0 & \mathbf{b}_2\cdot\dot{\mathbf{b}}_3 \\ \mathbf{b}_3\cdot\dot{\mathbf{b}}_1 & \mathbf{b}_3\cdot\dot{\mathbf{b}}_2 & 0 \end{pmatrix}, $$

where the diagonal elements are known to vanish for all orthonormal basis vectors, and as it should by antisymmetry of the matrix.

The prescription to construct a vector ${}^N\omega^B$, such that

$$ {}^N\Omega^B \cdot \mathbf{r} = {}^N\omega^B \times \mathbf{r} $$

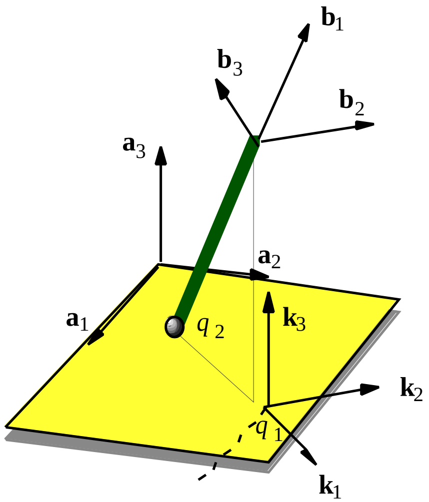
then yields

$$ {}^N\omega^B = b\left(\mathbf{b}_3 \cdot \dot{\mathbf{b}}_2 \quad \mathbf{b}_1 \cdot \dot{\mathbf{b}}_3 \quad \mathbf{b}_2 \cdot \dot{\mathbf{b}}_1\right)^T. $$

In particular, the vector ${}^N\omega^B$ corresponding to the angular velocity dyad is known as the **angular velocity vector**.

## Example: Calculation of angular velocities with SOPHIA

**This is the 'Illustration' problem in ML, page 46-48.**
**A pivoted rod is given. See figure. Find the angular velocity dyad, or the corresponding vector in the B-frame.**




—— SOPHIA ——

```
> restart;
read sophia21_3;
```

*sophia21_3 - 28 October 1997*

Warning: new definition for norm
Warning: new definition for trace

**Analyse simple rotations between frames**

```
> rotList:=[[A,K,3,q1],[K,B,2,-q2]];
 chainSimpRot(rotList):
```

$$ rotList := [[A, K, 3, q1], [K, B, 2, -q2]] $$

**Which parameters depend on time?**

```
>dependsTime(q1,q2):
```

**Alternative ways to calculate angular velocity between frames:**

```
> wAB:=angularVelocity(B,A);
uAB:= A &aV B;
```

$$ wAB := [[\sin(q2)\, q1t \quad -q2t \quad q1t\cos(q2)], B] $$

$$ uAB := [[\sin(q2)\, q1t \quad -q2t \quad q1t\cos(q2)], B] $$

**Express Evector in the B-frame:**

```
>BuAB:= B &to uAB;
```

$$ BuAB := [[\sin(q2)\, q1t \quad -q2t \quad q1t\cos(q2)], B] $$

**Express vector as a dyad (matrix):**

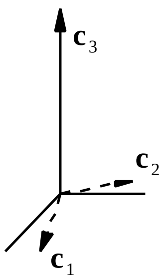
```
> WAB:=&VtoD(wAB);
```

$$ WAB := \left[\begin{bmatrix} 0 & -q1t\cos(q2) & -q2t \\ q1t\cos(q2) & 0 & -\sin(q2)\, q1t \\ q2t & \sin(q2)\, q1t & 0 \end{bmatrix}, B\right] $$

## Solved Problems

### P2.6

Find the angular velocity vector relating the fixed Cartesian and a moving <u>cylindrical</u> coordinate bases.




### Solution

As in **P1.2** we let the Cartesian coordinate system be spanned by the reference triad $n$ and introduce an auxiliary reference triad $c$ rotated relative to $n$ an amount $\varphi$ about the common direction $\mathbf{n}_3$. Then since this amounts to a simple rotation, the result is obviously

$$^{N}\omega^{C} = \dot{\varphi}\,\mathbf{n}_3.$$

### P2.7

Find the angular velocity vector relating the Cartesian and <u>spherical</u> coordinate bases.

### Solution

As in **P1.3** we let the Cartesian coordinate system be spanned by the reference triad $n$ and we introduce two auxiliary reference triads:

- $b$ rotated relative to $n$ an amount $\varphi$ about the common direction $\mathbf{n}_3$.
- $s$ rotated relative to $b$ an amount $\vartheta$ about the common direction $\mathbf{b}_2$.

Then, since both these rotations are simple rotations, their angular velocity vectors are given by

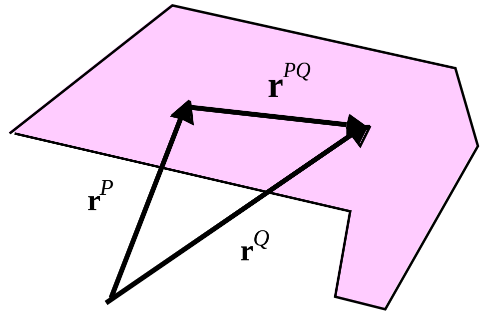
$$^{N}\omega^{B} = \dot{\varphi}\,\mathbf{n}_3, \quad \text{and} \quad ^{B}\omega^{S} = \dot{\vartheta}\,\mathbf{b}_2.$$

The additivity of angular velocity vectors then yields:

$$^{N}\omega^{S} = \dot{\varphi}\,\mathbf{n}_3 + \dot{\vartheta}\,\mathbf{b}_2.$$

## Velocities and frames

**P2.8** Let $P$ and $Q$ be two points fixed on a body, which in turn moves relative to a space fixed reference frame $N$. Find an expression for the difference between the **velocities** of $P$ and $Q$ relative to the reference frame $N$. Find also an expression for the difference between the **accelerations** of $P$ and $Q$ relative to the reference frame $N$.




**Solution** The position vectors corresponding to the points $P$ and $Q$ on the body relative to the reference point (origin) of the frame $N$ are related by (see figure): $\mathbf{r}^{Q} = \mathbf{r}^{P} + \mathbf{r}^{PQ}$,
where $\mathbf{r}^{PQ}$ is fixed in the 'body frame' $B$ with triad $b$. The velocity difference calculated in $N$ is expressed as:

$$\begin{aligned}
\mathbf{v}^{Q} - \mathbf{v}^{P} &= {}^{N}\frac{d}{dt}\mathbf{r}^{Q} - {}^{N}\frac{d}{dt}\mathbf{r}^{P} = {}^{N}\frac{d}{dt}\mathbf{r}^{PQ} \\
&= {}^{B}\frac{d}{dt}\mathbf{r}^{PQ} + {}^{N}\omega^{B} \times \mathbf{r}^{PQ} \\
&= {}^{N}\omega^{B} \times \mathbf{r}^{PQ}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbf{a}^{Q} - \mathbf{a}^{P} &= {}^{N}\frac{d}{dt}\mathbf{v}^{Q} - {}^{N}\frac{d}{dt}\mathbf{v}^{P} = {}^{N}\frac{d}{dt}\left({}^{N}\omega^{B} \times \mathbf{r}^{PQ}\right) \\
&= \left({}^{N}\frac{d}{dt}{}^{N}\omega^{B}\right) \times \mathbf{r}^{PQ} + {}^{N}\omega^{B} \times \left({}^{N}\omega^{B} \times \mathbf{r}^{PQ}\right),
\end{aligned}$$

since ${}^{N}\frac{d}{dt}\mathbf{r}^{PQ} = {}^{N}\omega^{B} \times \mathbf{r}^{PQ}$.

## P2.9

Find an expression relating derivatives of dyads in different reference frames.

### Solution

The partial derivative of a dyad $\mathbf{D}$ with repect to a variable $q$ relative to a reference frame $B$ is:

$${}^B\frac{\partial}{\partial q}\mathbf{D} = {}^B\frac{\partial}{\partial q}\left(b\,{}^bD\,b^T\right) = b\,\frac{\partial\,{}^bD}{\partial q}\,b^T,$$

whereas the corresponding derivative in the $A$ frame is:

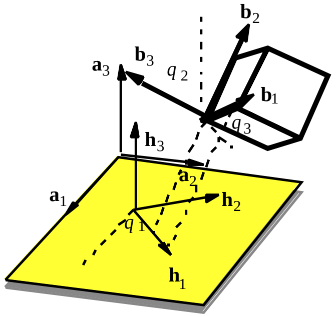
$$\begin{aligned}
{}^A\frac{\partial}{\partial q}\mathbf{D} &= {}^A\frac{\partial}{\partial q}\left(b\,{}^bD\,b^T\right)\\
&= \left({}^A\frac{\partial b}{\partial q}\right){}^bD\,b^T + b\,\frac{\partial\,{}^bD}{\partial q}\,b^T + b\,{}^bD\left({}^A\frac{\partial b}{\partial q}\right)^T\\
&= {}^B\frac{\partial \mathbf{D}}{\partial q} + \left({}^A\frac{\partial b}{\partial q}\,b^T\right)\cdot\mathbf{D} + \mathbf{D}\cdot\left(b\,{}^A\frac{\partial b^T}{\partial q}\right),
\end{aligned}$$

where we used the identity matrix $b^T\cdot b = \mathbf{1}$.

Identifying the angular rate dyads in the brackets, we finally get:

$${}^A\frac{\partial}{\partial q}\mathbf{D} = {}^B\frac{\partial \mathbf{D}}{\partial q} + {}^A\Omega_q^B\cdot\mathbf{D} - \mathbf{D}\cdot{}^A\Omega_q^B,$$

where the antisymmetry property of ${}^A\Omega_q^B$ is employed.




## P2.10

Find the angular velocity vector relating the Cartesian and a general frame defined by Euler angles, as a sequence of simple rotations.

### Solution

--------SOPHIA-------------------------------

**This is the 'Illustration' problem in ML, page 54-58.**

```
> restart;
read sophia21_3;
```

*sophia21_3 - 28 October 1997*

Warning: new definition for norm
Warning: new definition for trace

**Analyse simple rotations between frames:**

```
> rotList:=[[A,H,3,q1],[H,F,1,q2],[F,B,3,q3]];
chainSimpRot(rotList):
```

$$rotList := [[A, H, 3, q1], [H, F, 1, q2], [F, B, 3, q3]]$$

**Declare time-dependent parameters:**

```
> dependsTime(q1,q2,q3):
```

**Angular velocity vector in the A-frame:**

```
> wABA:= A &to (A &aV B);
```

$$wABA := [[\cos(q1)\,q2t + \sin(q1)\,\sin(q2)\,q3t,\ \sin(q1)\,q2t - \cos(q1)\,\sin(q2)\,q3t,\ q1t + \cos(q2)\,q3t], A]$$

**Angular acceleration vector calculated relative the A-frame and expressed in the H-frame:**

```
> aABA:= H &to (A &fdt wABA);
```

$$aABA := [[q2tt + q1t\,\sin(q2)\,q3t,\ q1t\,q2t - \cos(q2)\,q2t\,q3t - \sin(q2)\,q3tt,\ q1tt - \sin(q2)\,q2t\,q3t + \cos(q2)\,q3tt], H]$$