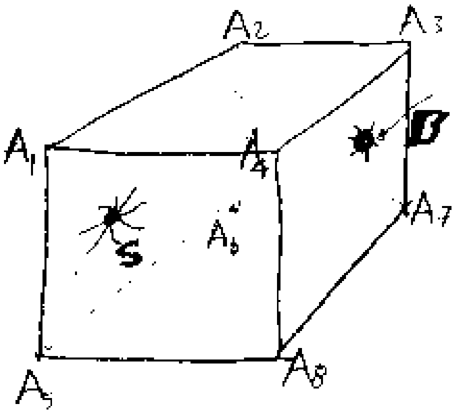
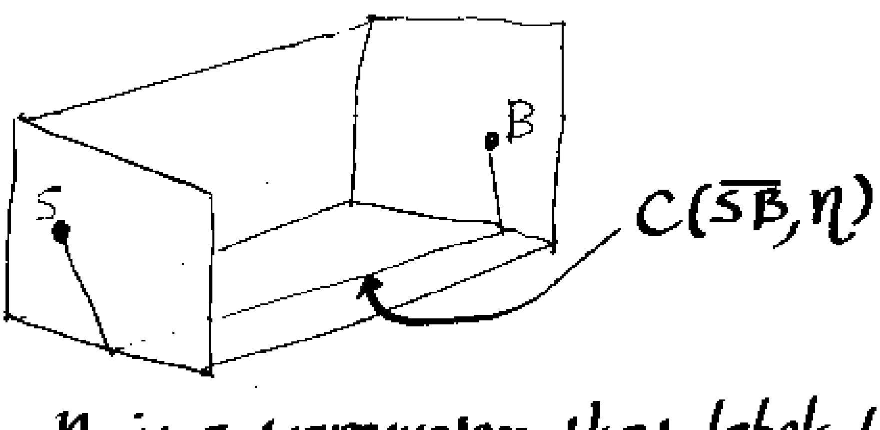
# The Calculus of Variations and Lagrange's Equations




A spider, in the interior of a box, on side $A_1A_4A_8A_5$ sees a juicy bug on side $A_2A_3A_7A_6$. Being a clever spider she calculates the shortest path to the bug. What is it?

This is a problem in the <u>Calculus of Variations</u>. There is a Distance Function $D_S$ that takes as arguments entire 'paths' - 'routes' from the spider $S$ to the bug $B$.

We can specify a path by its end points and a curve between the end points — (in this case we can assume the curve is a straight line).

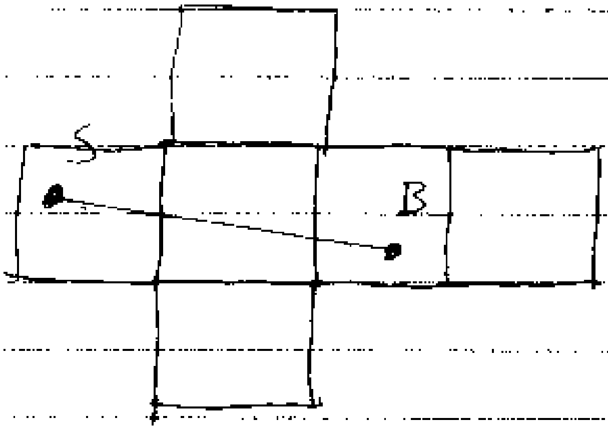



where $\eta$ is a parameter that labels the paths.

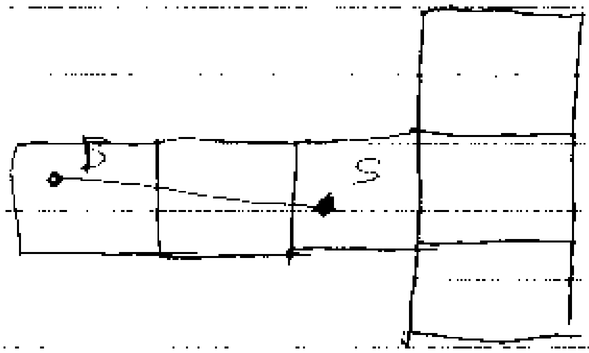
We want! a path $\bar{\eta}$ such that

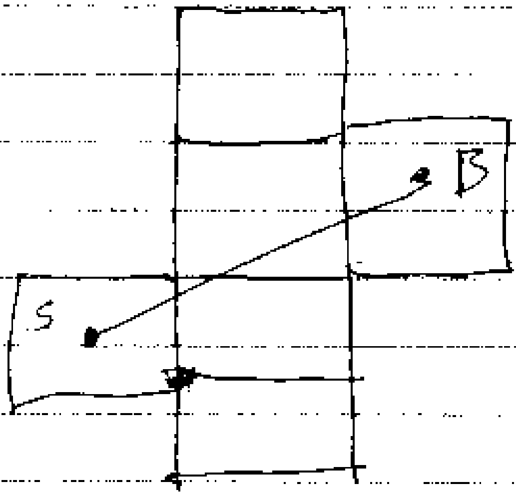
$$D_s\left[C(\bar{\eta})\right] \leq D_s\left[C(\eta)\right]$$

That is we want the shortest path. In this case we can find a graphical solution




In general there will be more then one path which is a "local" maximum or minimum. Just imagine the different ways you can unfold the box!




etc.

This is not the same as the calculus problem of finding maxima and minima.

The argument of the function is a <u>path</u> not a <u>point</u> or number.

We have instead a mapping from curves to numbers

$$\begin{cases} D_s[\ ]: & \text{curves} \longrightarrow \text{numbers} \\ & \text{paths} \longrightarrow \text{distances} \end{cases}$$

Also we are interesed in <u>local</u> maximum or minimum values — called <u>Stationary Values</u>.

General Mathematics Problem

$$I[c] = \int_{t_1}^{t_2} L(t, x(t), \dot{x}(t))\, dt$$

where $x(t_1)$, $x(t_2)$ fixed

C: $x(t)$. <u>curves</u>.


We want to find a curve $x(t)$ such that $I[C]$ is a local minimum. ~~point~~

for any curve we have $I[C]$ is a number.

$$I + \delta I = I[C + \delta C]$$

$$\delta I \cong \cancel{B}\ DI \cdot \delta C$$

we want to find $DI$ which is an operator on curves — i.e. it takes $x, \dot{x}$ and operates on $\delta C$ to produce a number. To make this concrete —

let $C = \{x, \dot{x}\}$ define the curve.

Define a "variation" by $\varepsilon\eta(t)$ where $\eta(t_1) = \eta(t_2) = 0$ and $|\varepsilon| < 1$

Then

$$I[C+\delta C] = I[C] + \delta I[C]$$

$$= \int_{t_1}^{t_2} L\left(t,\ x(t)+\varepsilon\eta(t),\ \dot{x}(t)+\varepsilon\dot{\eta}(t)\right)dt$$

now using Taylor's ~~[illegible]~~ expansion

$$L(t, x+\varepsilon\eta, \dot{x}+\varepsilon\dot{\eta}) = L(t,x,\dot{x}) + \varepsilon\left(\frac{\partial L(t,x,\dot{x})}{\partial x}\right)\eta$$

$$\varepsilon\left(\frac{\partial L(t,x,\dot{x})}{\partial \dot{x}}\right)\dot{\eta} + \cdots$$ higher order terms in $\varepsilon$.

hence

$$I + \delta I[C] = \overbrace{\int_{t_1}^{t_2} L(t,x,\dot{x})\,dt}^{I[C]} + \varepsilon\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial x}\eta + \frac{\partial L}{\partial \dot{x}}\dot{\eta}\right)dt$$

$$+ O(\varepsilon^2)$$

$$\boxed{\delta I[C] = \varepsilon\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial x}\eta + \frac{\partial L}{\partial \dot{x}}\dot{\eta}\right)dt + O(\varepsilon^2)}$$

We would like to obtain a form like

$$\delta I[c] = \int_{t_1}^{t_2} S(\varepsilon\eta)\,dt$$

but we have $\dot{\eta}$ present – how to get rid of $\dot{\eta}$?

Consider $d(UV) = (dU)V + U(dV)$

or $U\,dV = d(UV) - V\,dU$

let $U = \frac{\partial L}{\partial \dot{x}}$ and $dV = \dot{\eta}\,dt = \frac{d\eta}{dt}\,dt$

thus: $dU = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) dt$, $V = \eta$

So

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{x}}\,\dot{\eta}\,dt = \left[\eta(t)\frac{\partial L}{\partial \dot{x}}\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \eta\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right) dt$$

But $\eta(t_1) = \eta(t_2) = 0$

hence.

$$\delta I = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right) \varepsilon \eta(t)\, dt$$

$\varepsilon$ was an 'order' parameter so take and $\delta C = \varepsilon \eta$

$$\delta I[\varepsilon \eta] = \delta I[\delta C] = \int_{t_1}^{t_2} \mathcal{L}(\delta C)\, dt$$

where $\mathcal{L}(\delta C) = \left( \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right) \delta C$

We have a stationary point if

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

The argument remains the same if we have

$$L(q_1, \dot{q}_1, q_2, \dot{q}_2 \cdots)$$

~~whe~~ we then get

$$\boxed{\frac{\partial L}{\partial q_n} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_n} = 0}$$

For a free particle in two-d we have:

Kinetic Energy $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

and in a gravity field $-\hat{j}mg$.

$V = mgy$ , $-\frac{\partial V}{\partial y} = -mg$.

define L as. $T - V$ so.

$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$

$\frac{\partial L}{\partial x} = 0$ , $\frac{\partial L}{\partial y} = -mg$

$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$ $\quad \frac{\partial L}{\partial \dot{y}} = m\dot{y}$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{d}{dt}m\dot{x} = m\ddot{x} = 0$

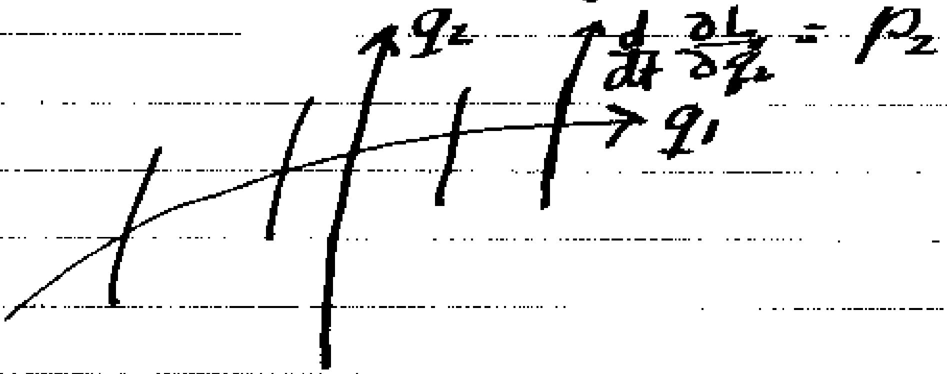
$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = \frac{d}{dt}m\dot{y} + mg. = 0$

$\therefore \left.\begin{aligned} m\ddot{x} &= 0 \\ m\ddot{y} &= -mg \end{aligned}\right\}$ as expected.

∴ for conservative systems the equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_n} - \frac{\partial L}{\partial q_n} = 0$$

where $q_n$ are all the coordinates needed to determine the configuration of the mechanical system.




we see $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_n}$ is the rate of change of momentum along the $q_2$ direction $q_1$ fixed.

$$\frac{\partial L}{\partial \dot{q}_n} = p_n = \text{generalized momentum.}$$

∴

$$\dot{p}_n = \left(\frac{\partial L}{\partial q_n}\right) \text{ are Lagranges equations}$$