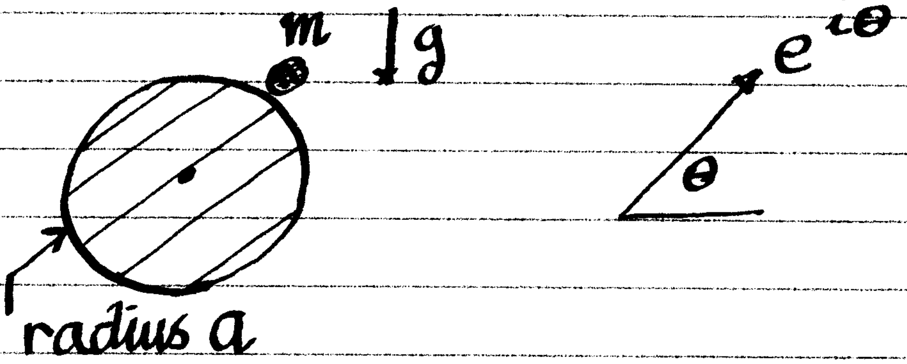
Particle on Cylinder Problem



The position of the particle is

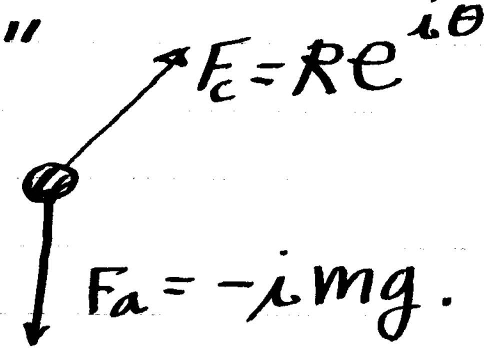
$z = ae^{i\theta}$, θ being measured from the horizontal so that $\theta = \pi/2$ at topmost position where $z = ae^{i\pi/2}$
$= ia$

The forces on the particle are

The "active" gravity force (weight)

$F_a = -img.$

The "constraint" force that prevents the mass from penetrating the cylinder which we can represent on the "Free body diagram"



Note that we "know" the constraint force direction is perpendicular to the cylinder surface. We do not know the magnitude R, which must be found as part of the solution!

Now compute the velocity (θ is a function of t, to be found)

$$\dot{z} = \frac{dz}{dt} = a\left(\frac{d}{d\theta}\left(e^{i\theta}\right)\right)\frac{d\theta}{dt}$$

by the chain rule!

or $$\boxed{\dot{z} = \left(i a e^{i\theta}\right)\dot{\theta}}$$

The instantaneous velocity direction is given by the coefficient of $\dot{\theta}$

Formally by $$\frac{\partial \dot{z}}{\partial \dot{\theta}} = i a e^{i\theta}$$

We call this the direction number associated with $\dot{\theta}$ or $Q_{\dot{\theta}}$ so

$$\boxed{Q_{\dot{\theta}} = i a e^{i\theta}}$$

The importance of this number is that it is perpendicular to the direction of the constraint force — Therefore it provides a means of removing that unknown from our final equation of motion. Formally we will use the fact that.

$$\text{Real}\left(Q^*_{\dot{\theta}} F_c\right) = 0$$

where $Q^*_{\dot{\theta}}$ is the complex conjugate
$= -i a e^{-i\theta}$

The complex momentum

$$p = m\dot{z} = i a m e^{i\theta}\dot{\theta}$$

We need the derivative of this $\to \dot{p}$ to form the equation of motion

$$\dot{p} = m\ddot{z}$$
$$= iam\ddot{\theta}e^{i\theta} - am\dot{\theta}^2 e^{i\theta}$$

The equation of motion is

$$\dot{p} - F_a - F_c = 0$$

or

$$iam\ddot{\theta}e^{i\theta} - am\dot{\theta}^2 e^{i\theta} + img - Re^{i\theta} = 0$$

Now muliply by $Q^*_{\dot{\theta}}$ to get

$$Q^*_{\dot{\theta}}(\dot{p} - F_a - F_c) = 0$$

or

$$0 = -iae^{-i\theta}(iam\ddot{\theta}e^{i\theta} - am\dot{\theta}^2 e^{i\theta} + img - Re^{i\theta})$$

or

$$ma^2\ddot{\theta} + ia^2m\dot{\theta}^2 + amge^{-i\theta} + iRa = 0$$

As planned the <u>real</u> part of this will be constraint force free!

Real (previous equation) $= 0$

or $\boxed{ma^2\ddot{\theta} + amg\cos\theta = 0}$

where we used $e^{-i\theta} = \cos\theta - i\sin\theta$

or finally

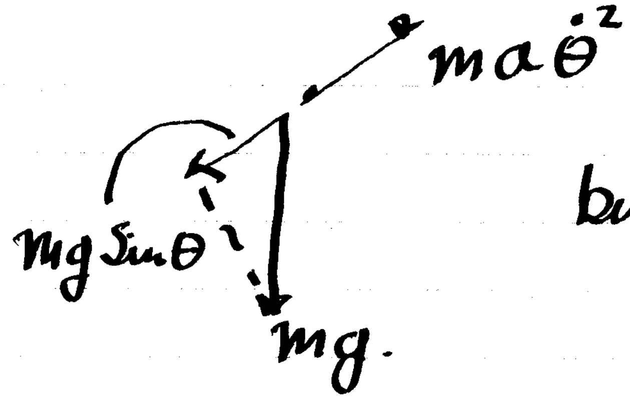
$$\boxed{\ddot{\theta} + \left(\frac{g}{a}\right)\cos\theta = 0}$$

Now the imaginary part of the last equation on page 4 gives an equation for R

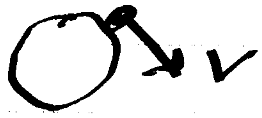
$$a^2m\dot{\theta}^2 \cancel{\theta} + Ra - amg\sin\theta = 0$$

or $R = \dfrac{amg\sin\theta - a^2m\dot{\theta}^2}{a}$

$$\boxed{R = mg\sin\theta - am\dot{\theta}^2}$$



but $a\dot{\theta} = V$



the tangential velocity.

$$ma\dot{\theta}^2 = ma\frac{V^2}{a^2} = \frac{mV^2}{a}$$

which is the centrif. force. !

if $R = 0 \Rightarrow \cancel{m}g\sin\theta = \frac{\cancel{m}V^2}{a}$

So the condition for leaving the cylinder is

$$V^2 = ag\sin\theta_c$$

θ_c = critical angle.