Advection-Diffusion equation:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$$

Finite-Difference approximation: $u_j^n = u(x_j, t^n)$

$$\frac{\partial u}{\partial t} = \frac{u_j^{n+1} - u_j^n}{\Delta t} + \mathcal{O}(\Delta t) \qquad \text{(Euler forward)}$$

$$\frac{\partial u}{\partial x} = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \mathcal{O}(\Delta x^2) \qquad \text{(central space)}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + \mathcal{O}(\Delta x^2) \qquad \text{(central space)}$$

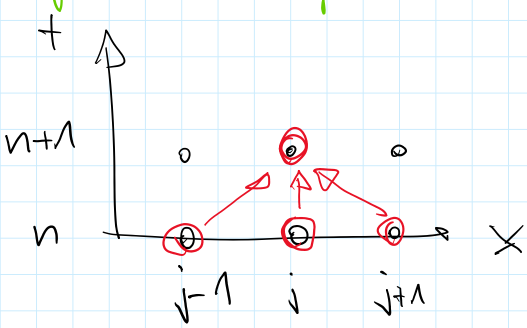
$$\Longrightarrow \frac{u_j^{n+1} - u_j^n}{\Delta t} + c\,\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = \nu\,\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

explicit method: RHS approximated at "old time" n

$$\Longrightarrow \boxed{u_j^{n+1} = u_j^n - c\frac{\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) + \nu\frac{\Delta t}{\Delta x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)}$$

update rule for each grid point

Stencil:



NB: consistent with boundary conditions mentioned earlier.

Question: What about stability?

- CFL condition for advection:

$$\sigma = c\frac{\Delta t}{\Delta x} \leq 1 \qquad \text{(necessary)}$$

- von Neumann analysis for diffusion

$$\beta = \nu \frac{\Delta t}{\Delta x^2} \leq 1/2 \qquad \text{(necessary \& sufficient)}$$

- coupled problem: (von Neumann analysis, homework)

$$\underbrace{\sigma^2 \leq \overbrace{2\beta \leq 1}}$$

$$\beta \leq 1/2 \Rightarrow \Delta t \leq \frac{1}{2\nu}\Delta x^2$$

$$\frac{\sigma^2}{\beta} \leq 2 \Rightarrow \Delta t \leq \frac{2\nu}{c^2}$$

Observations: for high ν: $\boxed{\Delta t \leq \frac{1}{2\nu}\Delta x^2}$

- "viscous limit"

for low ν: $\boxed{\Delta t \leq \frac{2\nu}{c^2}}$

both need to be fulfilled!

- independent of Δx.
- advection stabilized by viscosity. ($\nu = 0 \Rightarrow \Delta t = 0$)