Control of the linearized channel flow via adjoint-based iterative optimization

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1 Aim of the project

This project focuses on the adjoint-based optimal control of small disturbances in a channel flow. The linear evolution of these disturbances is governed by the Orr-Sommerfeld/Squire equations. The control is applied by means of a time dependent blowing and suction $a(t)$ at the lower wall ($y = -1$), cf. fig. 1. The objective of the controller is to minimize the disturbance energy in the domain at some time $t = T$. A measure of the control input is also included to regularize the optimization problem and bound the control energy used. So the objective function can be written as,

$$J = \frac{1}{2}\left(\frac{E(T)}{E(0)} + l^2 \int_0^T a^2(t)\mathrm{d}t\right), \tag{1}$$

where $E(t)$ is the disturbance energy $\int_0^t \mathbf{q}^H \mathbf{q} \mathrm{d}t$ at the time t and l is the control penalty. The optimization problem is then: find a^* which satisfies,

$$J(a^*) \leq J(a), \quad \forall a(t) \in \mathcal{A}, \tag{2}$$

where $\mathcal{A}$ denotes the set of admissible controls.

2 Governing equations

In the following, the governing theory of the optimal control strategy is presented, beginning with the flow equations. Subsequently the adjoint control approach is carried out. We refer to the book by Schmid and Henningson [1] and to the review paper by Bewley [2] for more thorough derivations.

2.1 Flow equations

The Orr-Sommerfeld and Squire equations can be expressed in a $v - \eta$ perturbation formulation with a time independent velocity-profile U,

$$\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 - \frac{d^2U}{dy^2}\frac{\partial}{\partial x} - \frac{1}{\mathrm{Re}}\nabla^4\right] v = 0. \tag{3}$$

Here, the centerline Reynolds number for channel flow is used, i.e. $\mathrm{Re} = U_c L/\nu$. A second equation for $\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$ is,

$$\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \frac{1}{\mathrm{Re}}\nabla^2\right]\eta = -\frac{dU}{dy}\frac{\partial v}{\partial z}. \tag{4}$$

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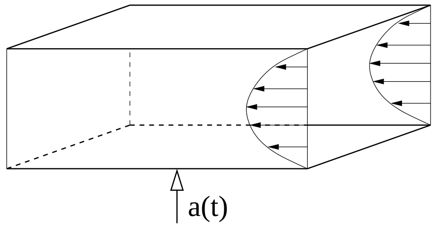

Figure 1: Sketch of the physical domain with the time dependent forcing (control) $a(t)$.

The above Orr-Sommerfeld/Squire equations are written in block form

$$\underbrace{\begin{bmatrix} \mathcal{B} & 0 \\ 0 & \mathcal{I} \end{bmatrix}}_{\tilde{\mathcal{B}}} \begin{pmatrix} \dot{v} \\ \dot{\eta} \end{pmatrix} = \underbrace{\begin{bmatrix} \mathcal{A} & 0 \\ \mathcal{LC} & \mathcal{M} \end{bmatrix}}_{\tilde{\mathcal{A}}} \begin{pmatrix} v \\ \eta \end{pmatrix} \tag{5}$$

After a Fourier decomposition in x and z of the type $\exp(\alpha x + \beta z)$, the Orr-Sommerfeld operator $\mathcal{B}, \mathcal{A}$ simplifies into

$$\mathcal{B} = \frac{\partial^2}{\partial y^2} - k^2, \tag{6}$$

$$\mathcal{A} = \alpha U'' - \alpha U\left(\frac{\partial^2}{\partial y^2} - k^2\right) - \frac{i}{\mathrm{Re}}\left(\frac{\partial^4}{\partial y^4} - 2k^2\frac{\partial^2}{\partial y^2} - k^2\right), \tag{7}$$

where $k^2 = \alpha^2 + \beta^2$. The Squire operator is simply

$$\mathcal{M} = \alpha U + i/\mathrm{Re}(\frac{\partial^2}{\partial y^2} - k^2). \tag{8}$$

Finally, the coupling operator is given by

$$\mathcal{LC} = -\beta U'. \tag{9}$$

The associated boundary conditions can be chosen as

$$\frac{\partial v}{\partial y}(-1) = \eta(-1) = 0,$$

$$v(+1) = \frac{\partial v}{\partial y}(1) = \eta(1) = 0.$$

In our case the control of the flow is achieved by a time-dependent blowing and suction at the lower wall which can be written as,

$$v(-1) = a(t). \tag{10}$$

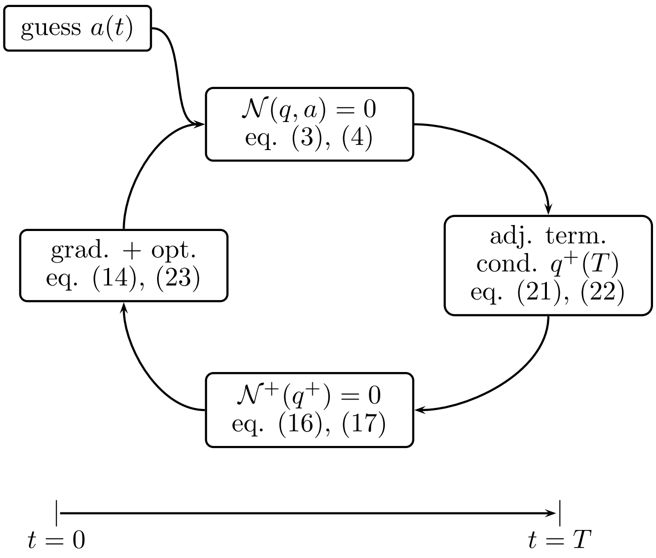

Figure 2: Iterative optimization loop.

2.2 Gradient evaluation using adjoint equations

In gradient based optimization, there are different ways to compute the gradients of interest. To solve the optimization problem formulated in the introduction, the gradient with respect to the control $a(t)$ can be defined through the directional derivate as,

$$\delta J = \lim_{s\to 0} \frac{J(a+s\delta a)-J(a)}{s}.$$

One way to solve a optimization problem is to perturb the control, solve the Orr-Sommerfeld/Squire equations and evaluate the objective function. A straightforward method to accomplish this is to use a finite difference approximation of the gradient,

$$\nabla J(a) \approx \frac{J(a+\epsilon)-J(a)}{\epsilon}.$$

However, the computational effort is very high when the dimension of controller, n, is large, i.e. the Orr-Sommerfeld/Squire equation have to be solved n times. Another approach, based on the adjoint equations can be used to evaluate the gradient of the objective function, which involves solving only two equations, the Orr-Sommerfeld/Squire equation and the adjoint Orr-Sommerfeld/Squire equation.

This framework can be formalised by introducing Lagrange multipliers, denoted as a^+ and q^+. These variables are also known as the adjoint variables of a and q, respectively. The Lagrangian can be written as:

$$L = J - \langle \tilde{\mathcal{B}}\dot{q} - \tilde{\mathcal{A}}q, q^+\rangle - (v(-1)-a, a^+), \qquad (11)$$

with the two scalar products defined as,

$$\langle p,q\rangle = \int_0^T\int_{-1}^1 pq\,\mathrm{d}y\mathrm{d}t, \qquad (12)$$

$$(p,q) = \int_0^T pq\,\mathrm{d}t. \qquad (13)$$

To minimize J subject to the Orr-Sommerfeld/Squire equation, we may equivalently minimize L with no constraints. If one derives the variation of the Lagrangian with respect to the state q, the control $a(t)$ and the adjoint state q^+ and control a^+, respectively one obtains a set of equations, called optimality conditions, that are only satisfied at minimum point of J.

The first variation of the Lagrangian with respect to the adjoint state q^+ and the adjoint control a^+ leads to the state equation and the constraint of the control, respectively. The first variation of the Lagrangian with respect to a leads to a first optimality condition which is also the gradient of J expressed only in terms of a and a^+,

$$\frac{\partial J}{\partial a} = -a^+ + 2l^2 a. \qquad (14)$$

To evaluate $\frac{\partial J}{\partial a}$, a second optimality condition, resulting from the integration by parts, links the adjoint variables q^+ and a^+ over a third derivative with respect to y at the lower wall,

$$a^+ = -1/Re\frac{\partial^3 v^+}{\partial y^3}. \qquad (15)$$

The adjoint state q^+ is obtained by solving the adjoint Orr-Sommerfeld equations with appropriate boundary conditions and initial conditions. Observe that the initial conditions of adjoint state are called terminal conditions, because they depend on the state at time $t=T$. These equations result from the first variation of the Lagrangian with respect to the state q,

$$\left(\frac{\partial^2}{\partial y^2}-k^2\right)\dot{v}^+ + \alpha U\left(\frac{\partial^2}{\partial y^2}-k^2\right)v^+ + 2U'\frac{\partial v^+}{\partial y} - i/\mathrm{Re}\left(k^4 - 2k^2\frac{\partial^2}{\partial y^2}+\frac{\partial^4}{\partial y^4}\right)v^+ + \beta U'\eta^+ = 0 \qquad (16)$$

and

$$\dot{\eta}^+ + \alpha U\eta^+ - i/Re\left(\frac{\partial^2}{\partial y^2}-k^2\right)\eta^+ = 0. \qquad (17)$$

Integration by parts leads to the adjoint boundary conditions,

$$v^+(-1) = v^+(1) = 0, \qquad (18)$$

$$\frac{\partial v^+}{\partial y}(-1) = \frac{\partial v^+}{\partial y}(+1) = 0, \qquad (19)$$

$$\eta^+(-1) = \eta^+(1) = 0. \qquad (20)$$

and the adjoint terminal conditions,

$$\left(\frac{\partial^2}{\partial y^2}-k^2\right)v^+(T) = -\frac{1}{k^2E(0)}\left(\frac{\partial^2}{\partial y^2}-k^2\right)v(T). \qquad (21)$$

Note, that one has to solve the above Poisson equation to get the terminal condition in v^+, where the different boundary conditions of the direct and adjoint state have to be taken into account, carefully. In η^+, the terminal condition simply yields:

$$\eta^+(T,y) = -\frac{1}{k^2E(0)}\eta(T,y) \qquad (22)$$

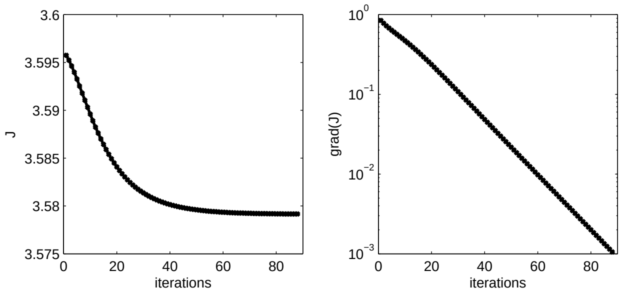

Figure 3: *Left:* objective function J. *Right:* gradient of the objective function J with respect to the control a.

The optimal control $a^*(t)$ can be computed in an iterative manner, cf. fig. 2. Beginning with an initial guess of the control one solves the direct Orr-Sommerfeld/Squire equations forward in time. At time $t = T$ the adjoint terminal conditions can be specified to compute the adjoint state backwards in time from $t = T$ to $t = 0$. Now the gradient and the new, optimized control can be computed:

$$a_{n+1} = a_n - \frac{\partial J}{\partial a} \cdot s \tag{23}$$

with a predefined step-size s.

3 Results, Conclusions and Further Work

We apply the adjoint method to minimize the objective function for one wavenumber pair, $\alpha = 1, \beta = 1$ and Re $= 1000$. The terminal time was chosen and $T = 12.5$. The Orr-Sommerfeld/Squire equation are discretized in space with second-order finite differences in wall-normal direction and in time with Matlab subroutine, *ODE45*, is used. The initial guess of state is a disturbance optimized to have the largest energy growth at the terminal time T. The above optimization procedure was applied with two different initial guesses of the control, $a(t) = 0$ and $a(0) = 0.5\sin(0.08\pi t)$, both converging towards the same minimum of the objective function. In Figure 3 the values of J and its gradient ∇J are shown. The minimum is found after 80 iterations, where the gradient is of the order $\nabla J < 10E - 3$.

Further work could be to test the performance of the above controller. Also, the next natural step is to use a more advanced optimization method than the steepest descent, e.g. conjugate gradient or BFGS methods.

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Bibliography

[1] P.J. Schmid and D. Henningson Stability and Transition in Shear Flows *Springer*, 2001

[2] T.R. Bewley Flow control: new challenges for a new renaissance. *Prog. Aerospace Sci.*, 37:21-58