Control of the linearized channel flow via adjoint-based iterative optimization

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1 Aim of the project
This project focuses on the adjoint-based optimal control of small disturbances in a channel flow. The linear evolution of these disturbances is governed by the Orr-Sommerfeld/Squire equations. The control is applied by means of a time dependent blowing and suction at the lower wall \( y = -1 \), cf. fig. 1. The objective of the controller is to minimize the disturbance energy in the domain at some time \( t = T \). A measure of the control input is also included to regularize the optimization problem and bound the control energy used. So the objective function can be written as,

\[
J = \frac{1}{2} \left( \frac{E(T)}{E(0)} + t^2 \int_0^T a^2(t)dt \right),
\]

where \( E(t) \) is the disturbance energy \( \int_0^T q^H q dt \) at the time \( t \) and \( l \) is the control penalty. The optimization problem is then: find \( a^* \) which satisfies,

\[
J(a^*) \leq J(a), \quad \forall a(t) \in A,
\]

where \( A \) denotes the set of admissible controls.

2 Governing equations
In the following, the governing theory of the optimal control strategy is presented, beginning with the flow equations. Subsequently the adjoint control approach is carried out. We refer to the book by Schmid and Henningson [1] and to the review paper by Bewley [2] for more thorough derivations.

2.1 Flow equations
The Orr-Sommerfeld and Squire equations can be expressed in a \( v - \eta \) perturbation formulation with a time independent velocity-profile \( U \),

\[
\left[ \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \nabla^2 - \frac{\partial^2 U}{\partial y^2} \frac{\partial}{\partial x} - \frac{1}{\text{Re}} \nabla^4 \right] v = 0.
\]

Here, the centerline Reynolds number for channel flow is used, i.e. \( \text{Re} = U' L / \nu \). A second equation for \( \eta = \frac{\partial u}{\partial x} - \frac{\partial w}{\partial z} \) is,

\[
\left[ \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} - \frac{1}{\text{Re}} \nabla^2 \right] \eta = -\frac{dU}{dy} \frac{\partial v}{\partial z} \]

\[
(4)
\]

The above Orr-Sommerfeld/Squire equations are written in block form

\[
\begin{bmatrix}
B & 0 & T \\
0 & \tilde{\xi}
\end{bmatrix}
\begin{bmatrix}
\dot{v} \\
\dot{\eta}
\end{bmatrix} = \begin{bmatrix}
A & 0
\end{bmatrix}
\begin{bmatrix}
\mathcal{L} C \\
M
\end{bmatrix}
\begin{bmatrix}
v \\
\eta
\end{bmatrix}
\]

After a Fourier decomposition in \( x \) and \( z \) of the type \( \exp(\alpha x + \beta z) \), the Orr-Sommerfeld operator \( B, A \) simplifies into

\[
B = \frac{\partial^2}{\partial y^2} - k^2,
\]

\[
A = \alpha U'' - \alpha U \left( \frac{\partial^2}{\partial y^2} - k^2 \right)
- \frac{i}{\text{Re}} \left( \frac{\partial^4}{\partial y^4} - 2k^2 \frac{\partial^2}{\partial y^2} - k^2 \right),
\]

where \( k^2 = \alpha^2 + \beta^2 \). The Squire operator is simply

\[
\mathcal{M} = \alpha U + i/\text{Re}(\frac{\partial^2}{\partial y^2} - k^2).
\]

Finally, the coupling operator is given by

\[
\mathcal{L} C = -\beta U'.
\]

The associated boundary conditions can be chosen as

\[
\frac{\partial v}{\partial y}(-1) = \eta(-1) = 0,
\]

\[
v(+1) = \frac{\partial v}{\partial y}(1) = \eta(1) = 0.
\]

In our case the control of the flow is achieved by a time-dependent blowing and suction at the lower wall which can be written as,

\[
v(-1) = a(t).
\]

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To minimize $J$ subject to the Orr-Sommerfeld/Squire equation, we may equivalently minimize $L$ with no constraints. If one derives the variation of the Lagrangian with respect to the state $q$, the control $a(t)$ and the adjoint state $q^+$ and control $a^+$, respectively, one obtains a set of equations, called optimality conditions, that are only satisfied at minimum point of $J$.

The first variation of the Lagrangian with respect to the adjoint state $q^+$ and the adjoint control $a^+$ leads to the state equation and the constraint of the control, respectively. The first variation of the Lagrangian with respect to $a$ leads to a first optimality condition which is also the gradient of $J$ expressed only in terms of $a$ and $a^+$,

$$\frac{\partial J}{\partial a} = -a^+ + 2I^2a. \tag{14}$$

To evaluate $\frac{\partial J}{\partial q}$, a second optimality condition, resulting from the integration by parts, links the adjoint variables $q^+$ and $a^+$ over a third derivative with respect to $y$ at the lower wall,

$$a^+ = -1/Re \frac{\partial^3 v^+}{\partial y^3}. \tag{15}$$

The adjoint state $q^+$ is obtained by solving the adjoint Orr-Sommerfeld equations with appropriate boundary conditions and initial conditions.

The adjoint state $q^+$ is obtained by solving the adjoint Orr-Sommerfeld/Squire equations with appropriate boundary conditions and initial conditions. Observe that the initial conditions of adjoint state are called terminal conditions, because they depend on the state at time $t = T$. These equations result from the first variation of the Lagrangian with respect to the state $q$,

$$\left( \frac{\partial^2}{\partial y^2} - k^2 \right) \hat{v}^+ + \alpha U \left( \frac{\partial^2}{\partial y^2} - k^2 \right) v^+ + 2U' \frac{\partial v^+}{\partial y}$$

$$-i/Re \left( k^4 - 2k^2 \frac{\partial^2}{\partial y^2} + \frac{\partial^4}{\partial y^4} \right) v^+ + \beta U' \eta^+ = 0 \tag{16}$$

and

$$\hat{\eta}^+ + \alpha U \eta^+ - i/Re \left( \frac{\partial^2}{\partial y^2} - k^2 \right) \eta^+ = 0. \tag{17}$$

Integration by parts leads to the adjoint boundary conditions,

$$v^+(1) = v^+(0) \tag{18}$$

$$\frac{\partial v^+}{\partial y}(-1) = \frac{\partial v^+}{\partial y}(1) = 0, \tag{19}$$

$$\eta^+(-1) = \eta^+(1) = 0. \tag{20}$$

and the adjoint terminal conditions,

$$\left( \frac{\partial^2}{\partial y^2} - k^2 \right) v^+(T) = -\frac{1}{k^2 E(0)} \left( \frac{\partial^2}{\partial y^2} - k^2 \right) v(T). \tag{21}$$

Note, that one has to solve the above Poisson equation to get the terminal condition in $v^+$, where the different boundary conditions of the direct and adjoint state have to be taken into account, carefully.

In $\eta^+$, the terminal condition simply yields:

$$\eta^+(T, y) = -\frac{1}{k^2 E(0)} \eta(T, y). \tag{22}$$
The optimal control $a^*(t)$ can be computed in an iterative manner, cf. fig. 2. Beginning with an initial guess of the control one solves the direct Orr-Sommerfeld/Squire equations forward in time. At time $t = T$ the adjoint terminal conditions can be specified to compute the adjoint state backwards in time from $t = T$ to $t = 0$. Now the gradient and the new, optimized control can be computed:

$$a_{n+1} = a_n - \frac{\partial J}{\partial a} \cdot s$$

with a predefined step-size $s$.

### 3 Results, Conclusions and Further Work

We apply the adjoint method to minimize the objective function for one wavenumber pair, $\alpha = 1$, $\beta = 1$ and $Re = 1000$. The terminal time was chosen and $T = 12.5$. The Orr-Sommerfeld/Squire equation are discretized in space with second-order finite differences in wall-normal direction and in time with Matlab subroutine, $ODE45$, is used. The initial guess of state is a disturbance optimized to have the largest energy growth at the terminal time $T$. The above optimization procedure was applied with two different initial guesses of the control, $a(t) = 0$ and $a(0) = 0.5 \sin(0.08\pi t)$, both converging towards the same minimum of the objective function. In Figure 3 the values of $J$ and its gradient $\nabla J$ are shown. The minimum is found after 80 iterations, where the gradient is of the order $\nabla J < 10E - 3$.

Further work could be to test the performance of the above controller. Also, the next natural step is to use a more advanced optimization method than the steepest descent, e.g. conjugate gradient or BFGS methods.

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### Bibliography
