Model reduction and feedback control of the Blasius boundary-layer

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APS/DFD07 Meeting
Salt-Lake city, Utah, USA
Nov 18-20, 2007
Motivation and aim of investigation

- Discretized Navier-Stokes equations too complex for modern control design ($H_2/H_\infty$)
  - $n > 10^5$ degrees of freedom

- Model reduction: approximate the high-dimensional system with a low-dimensional system
  - $m < 100$ degrees of freedom
  - How can we find a low-dimensional model that describes the input-output behaviour?
    ➡️ balanced truncation

- **Objective**: Design a reduced feedback controller ($H_2/H_\infty$) to reduce the growth of linear perturbations and eventually delay transition to turbulence.
State-space formulation of Navier-Stokes

Linearized Navier-Stokes in state-space formulation:

\[
\dot{q} = Aq + Bu \quad q(0) = 0
\]
\[
y = Cq
\]

For control design must analyze:
- Perturbations → Cost function
- Perturbations → Sensors
- Actuators → Cost function
- Actuators → Sensors

Laplace transform to frequency domain:

\[
y = \underbrace{C(sI - A)^{-1} Bu}_{G(s)}
\]
Input-output behavior of 2D Blasius flow

Frequency response

Impulse response

- DNS: \( n=10^5 \)
- ROM: \( m=50 \)
Snapshot-based balanced truncation

Snapshots of direct simulation:
\[ X = [q(t_1) \ldots q(t_m)]\Delta_t \]

Snapshots of adjoint simulation:
\[ Y = [q^+(t_1) \ldots q^+(t_m)]\Delta_t \]

One SVD of small system
(Rowley 2005)
\[ Y^T X = U\Sigma V^T \]
Balanced modes as expansion basis

- One SVD of small system: \( Y^T X = U \Sigma V^T \)

- Balanced modes and adjoint balanced modes:
  \[
  T = [T_1 \ldots T_m] = XV\Sigma^{-1/2}
  \]
  \[
  S = [S_1 \ldots S_m] = YU\Sigma^{-1/2}
  \]

- Projection on balanced modes to obtain reduced system:
  \[
  \dot{\hat{q}} = S^T AT\hat{q} + S^T Bu
  \]
  \[
  y = CT\hat{q}
  \]
Model reduction error

- The reduced model
  - $m = 2$ has its peak value for the same frequency as the full model.
  - $m = 50$ gives nearly the exact frequency response.

- Fast decay of model reduction error $\|G - G_r\|_\infty$
Optimal feedback control – $H_2$

- **$H_2$ Problem:**
  - Find a control signal $u(t)$ based on the measurements $y(t)$ such that the influence of external disturbances $w(t)$ and $g(t)$ on the output $z(t)$ is minimized.

- **Control Objective:**
  \[
  J = \|z\|_2^2 = \int_0^T q^T C_1^T C_1 q + l^2 u^T u \, dt
  \]
Closed-loop: harmonic forcing

- Control design parameters:
  - control penalty $l = 10^{-4}$
  - noise contamination $\alpha = 0.01$

- The closed-loop:
  - dampens the dominant frequencies
  - amplifies certain high frequencies.
Closed-loop: stochastic forcing
Conclusions and outlook

- The reduced balanced system with less than 50 dimensions captures in the input-output behavior of the 2D flat plate with 100,000 dimensions.

- The reduced model can be used to design modern control strategies, such as $H_2$ and $H_\infty$ to efficiently damp the linear growth of disturbances.

- Extend to 3D Blasius boundary layer.

- Realistic actuators (blowing and suction at the wall), sensors (skin friction), disturbances (free-stream turbulence)...

- Delay transition to turbulence and perform experiments!