Perfomance of reduced-order models of fluid systems

Examplified on Blasius flow and Ginzburg-Landau equation

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Motivation

- Navier-Stokes equations too complex to manipulate
  - $n>10^6$ degrees of freedom

- Model reduction: approximate the high-dimensional system with a low-dimensional system
  - *m¼10* degrees of freedom

- Model reduction is problem dependent:
  - Each application has a suitable low-dimensional model that captures the essential feature of its dynamics.

- In control design one describes the relation between inputs-outputs in terms of transfer functions.
  - How can we find a low-dimensional model that describes the input-output behaviour?
  - Answer: Balanced truncation

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What is relevant for capturing the input-output behaviour?

- For control design we must analyze the mappings:
  - Perturbations $\longrightarrow$ Objective function
  - Perturbations $\longrightarrow$ Sensors
  - Actuators $\longrightarrow$ Objective function
  - Actuators $\longrightarrow$ Sensors

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- Write linearized Navier-Stokes in state-space formulation:

$$\begin{aligned} \dot{q} &= Aq + Bu \quad q(0) = q_0 \\ y &= Cq \end{aligned}$$

- The formal solution:

$$y = Ce^{At}q_0 + C\int_0^t e^{A(t-\tau)}u(x,\tau)\mathrm{d}\tau$$

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Find the components of the state that are easily excited by input

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Find the components of state with large influence on output

Study cases – Choice of A

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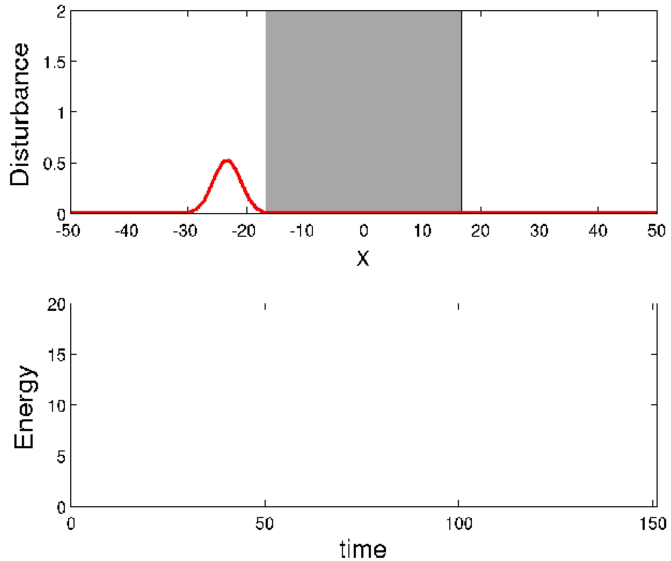
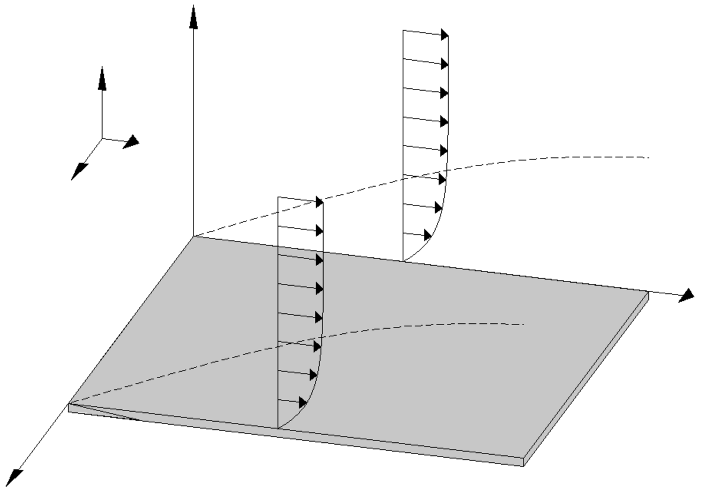
- 1D-case: Linear Ginzburg-Landau equation:
  - Convective and absolute instability
  - Transient growth due to non-normality of A



- 2D-case: Flat-plate boundary Layer -Blasius:
  - Re_{δ^*} = 500
  - n = 2(512x96) ¼ 10000
  - Spectral Code



States easily excited by input

For unit impulse input $u = \delta(t)$the state is given by,

$$q(t) = e^{At}B$$

we can measure the “size” of the state with the n£n matrix

$$X = \int_0^\infty q(t)q(t)^H \mathrm{d}t = \int_0^\infty e^{A\tau}BB^H e^{A^H\tau}\mathrm{d}\tau$$

This matrix is called the controllability Gramian.

- Gives a measure of the controllability of the components of a state
- Let T diagonalize X:

$$TXT^T = \mathrm{diag}\{\sigma_1, \ldots, \sigma_n\}$$

- Eigenvalue σ_i measures how much the state $T_i q$ is excited by the input
- Cannot neglect states corresponding to small σ_i, these state may have a large influence on the output!!

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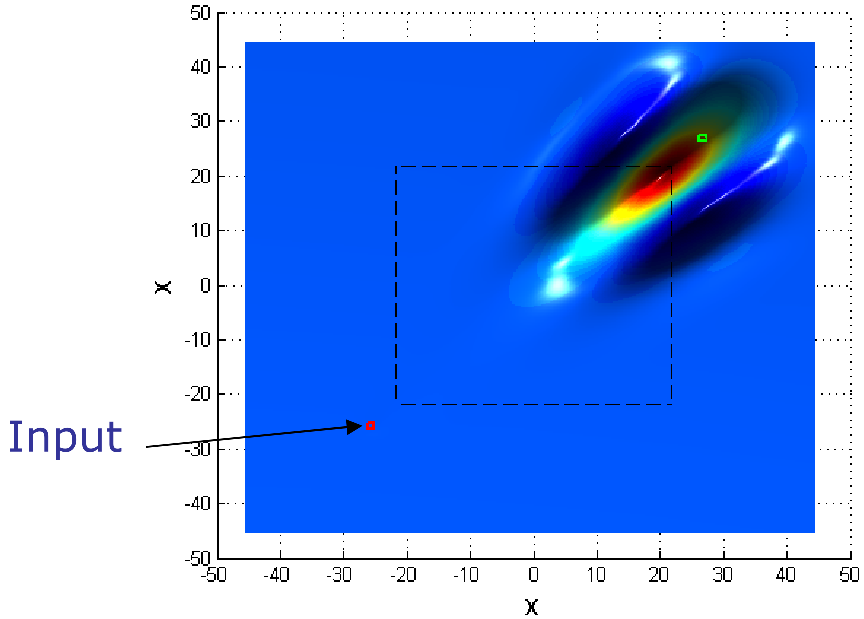
Controllability of Ginzburg-Landau



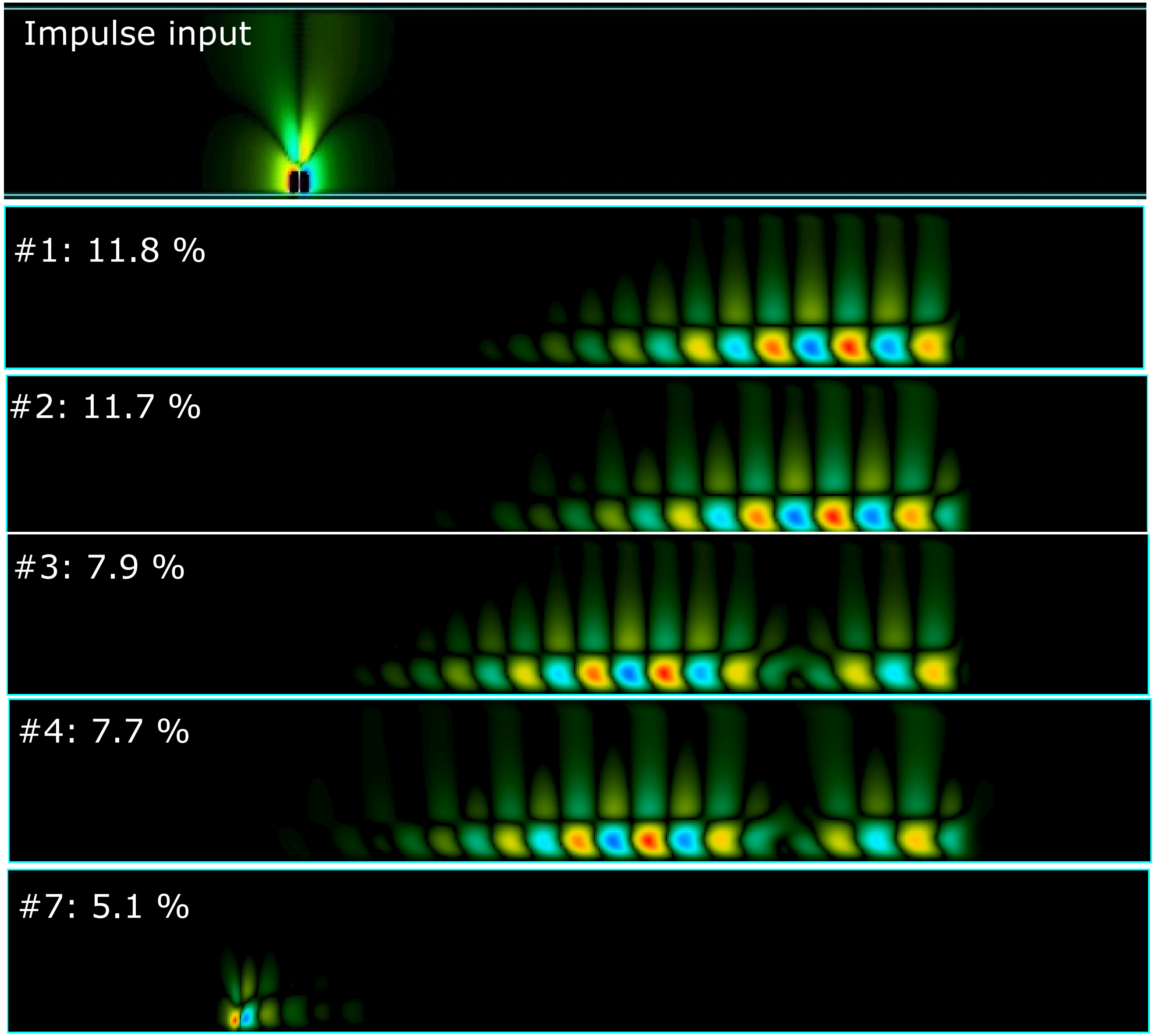
$$X = \underbrace{\int_0^{\infty} q(t)q(t)^H \mathrm{d}t}_{\text{State Covariance matrix}} = \underbrace{\int_0^{t} e^{A\tau}BB^H e^{A^H\tau} \mathrm{d}\tau}_{\text{Controllability Gramian}}$$

State Covariance matrix

- T are POD modes!
- Snapshot method

Controllability Gramian

Controllable modes of Blasius flow



400 snapshots

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States with large influence on output

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How much does a state contribute to the output?

Consider an initial state q_0, the state is give by

$$y(t) = Ce^{At} q_0$$

We can measure the size of the output by,

$$\|y\|^2 = q_0^H \underbrace{\int_0^\infty e^{A^H t} C^H C e^{At} \mathrm{d}t}_{Y} \, q_0 = q_0^H Y q_0$$

where Y, is called the observability Gramian.

- Measures how much energy a component of q_0 is transferred to output
- Let T^{-1} diagonalize Y:

$$T^{-T} Y T^{-1} = \mathrm{diag}\{\sigma_1, \ldots, \sigma_n\}$$

- If σ_i is zero, the output cannot sense $T^{-1}_i q_0$.

Observability of Ginzburg-Landau

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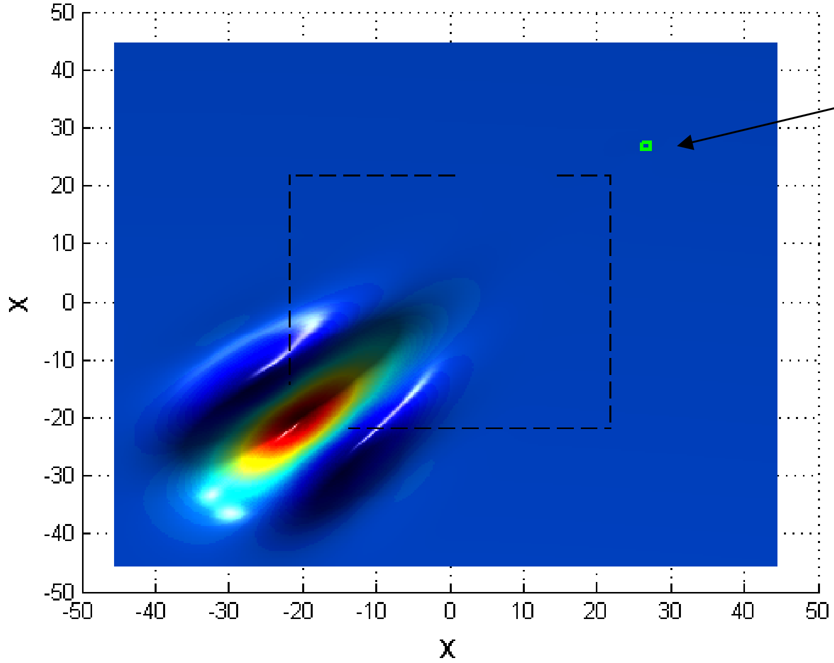


Output

$$\dot{q}^+ = A^H q^+ + C^H v$$

$$q^+ = e^{A^H t} C^H$$

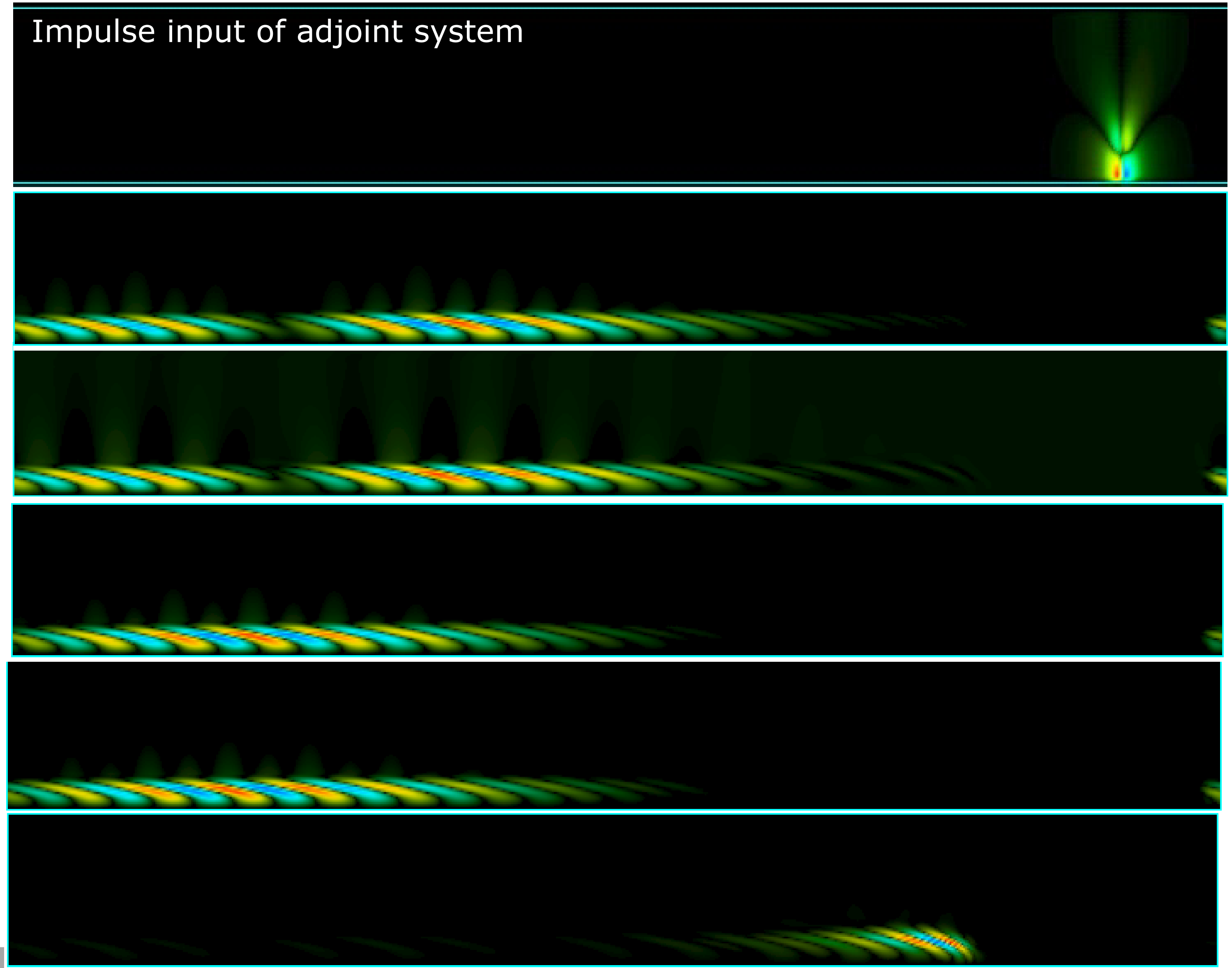
$$Y = \underbrace{\int_0^\infty e^{A^H t} C^H C e^{At} \mathrm{d}t}$$

Observability Gramain

Covariance matrix of adjoint state

- T^{-1} are adjoint POD modes!
- Snapshot method

Observable modes of Blasius flow



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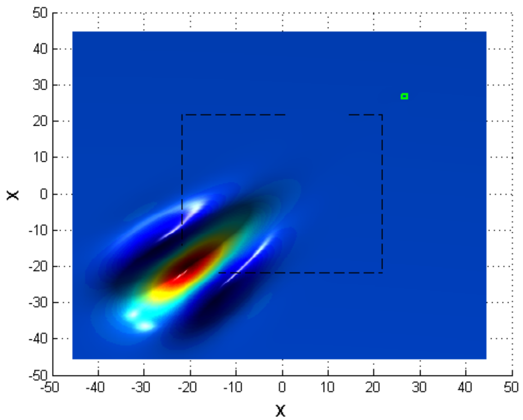
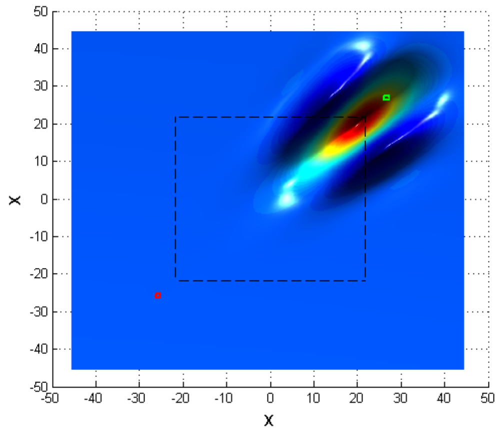
Balanced Realization

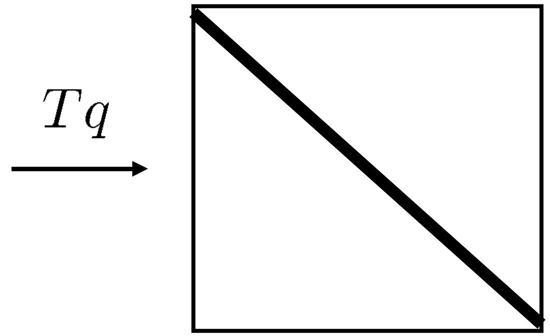
Unobservable and uncontrollable states do no influence the input-output behaviour of a system



Observable states



Controllable states



Balanced system

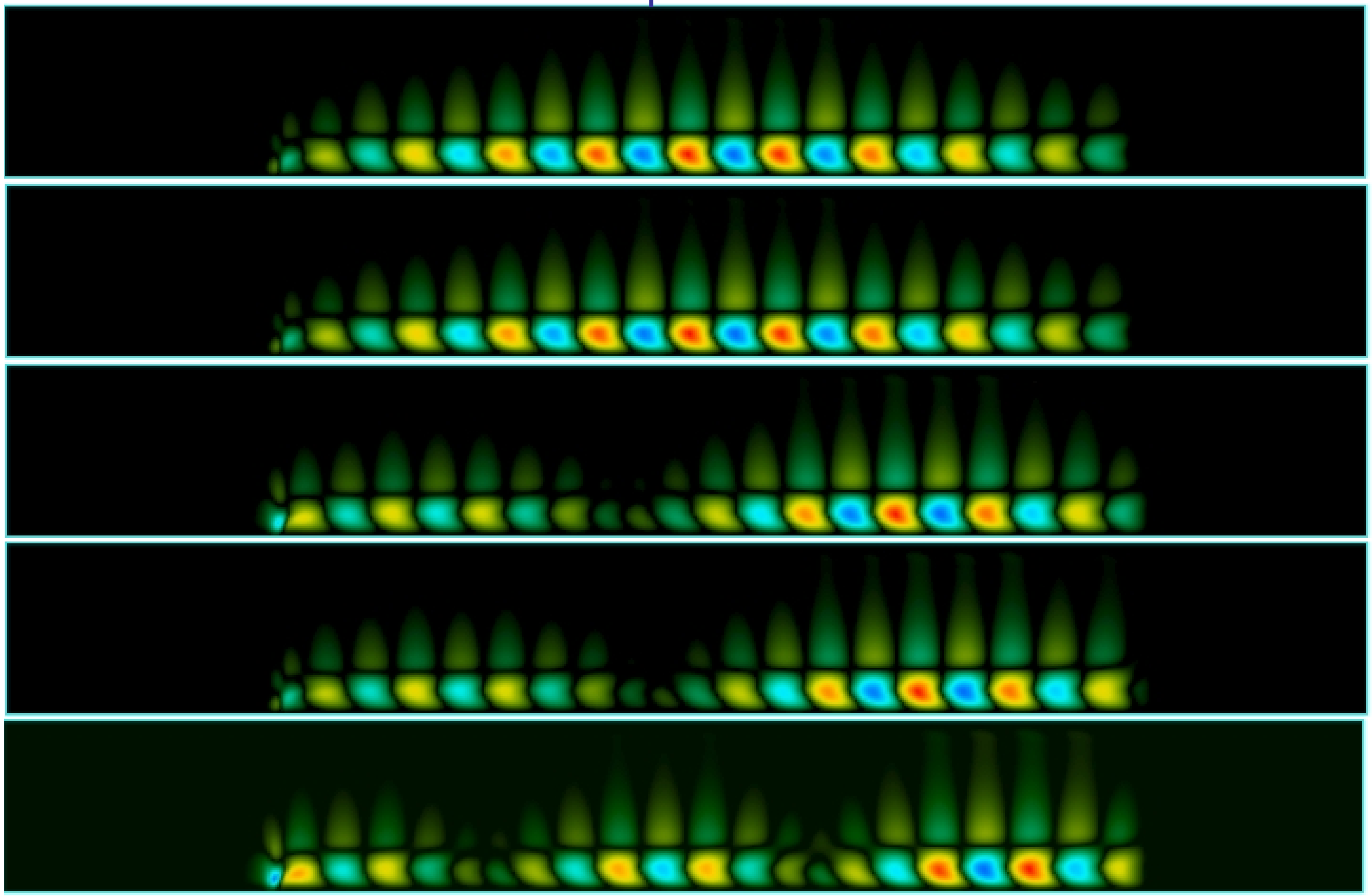
Choose T and T^{-1} so that the two Gramians become equal and diagonal:

$$TXT^T = T^{-T}YT^{-1} = \mathrm{diag}\{\sigma_1, \dots, \sigma_m\}$$

The columns of T contain the Balanced modes.

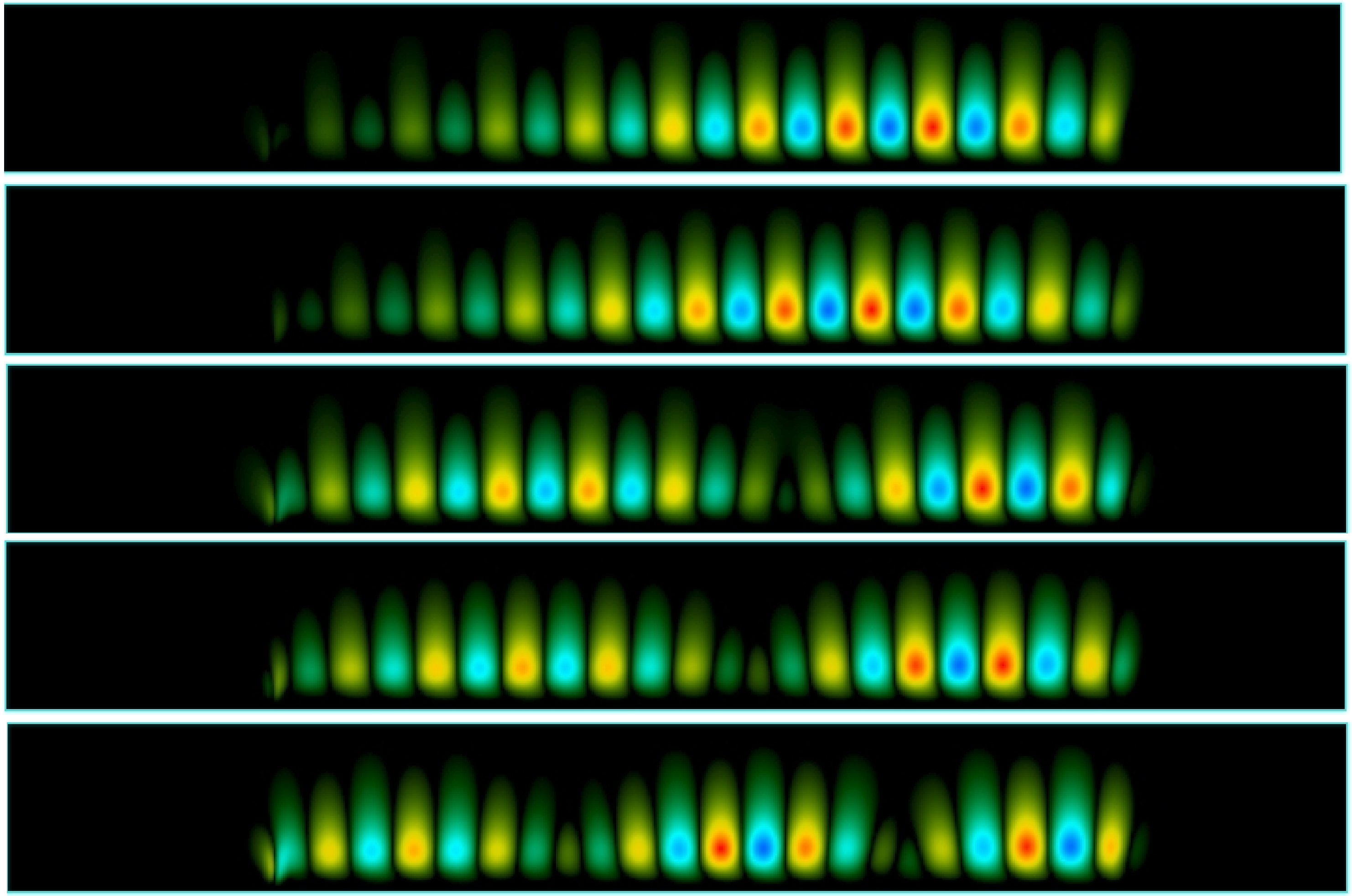
Balanced modes of Blasius flow

u-component



Balanced modes of Blasius flow

v-component



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Balanced truncation – model reduction

1. Compute balanced modes from snapshots

2. Change coordinates:

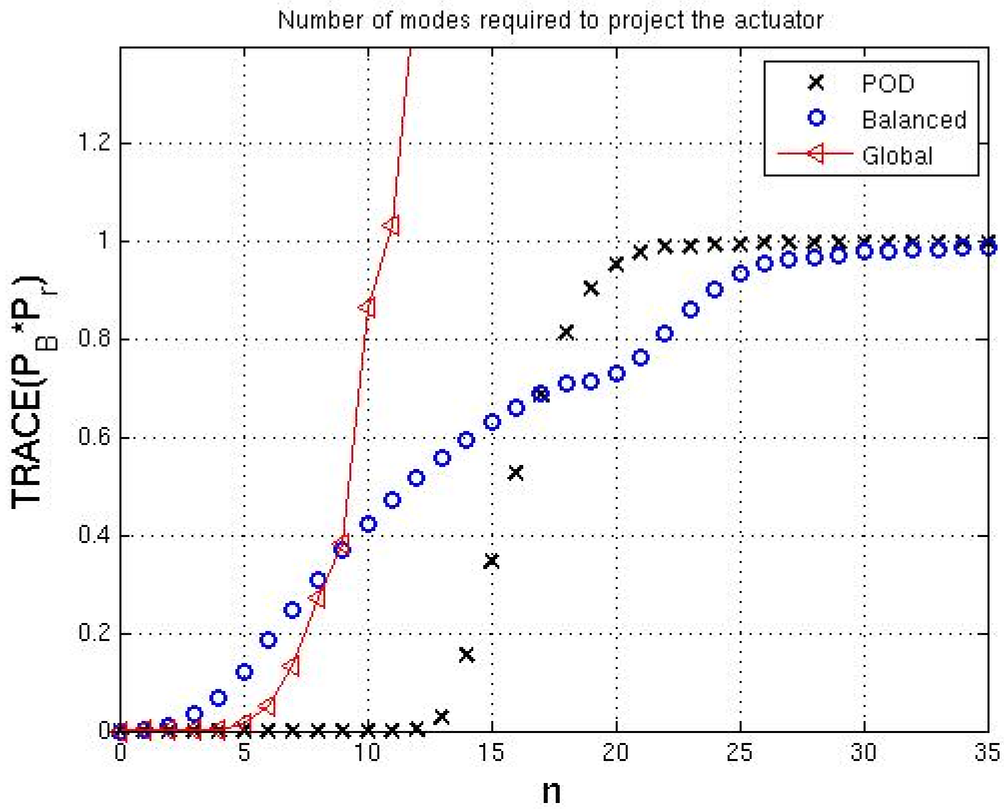
$$\hat{q} = Tq$$

3. Truncate the least controllable and observable states

$$\begin{aligned} \dot{q} &= Aq + Bu \\ y &= Cq \end{aligned} \quad \longrightarrow \quad \begin{aligned} \dot{\hat{q}} &= TAT^{-1}\hat{q} + TBu \\ y &= CT^{-1}q \end{aligned}$$

Huge   Small

Projection of actuator



- Number of modes required to project an actuator
- Actuator placed upstream, close to Branch I
- Reduced-order model of order 13, based POD captures nothing!

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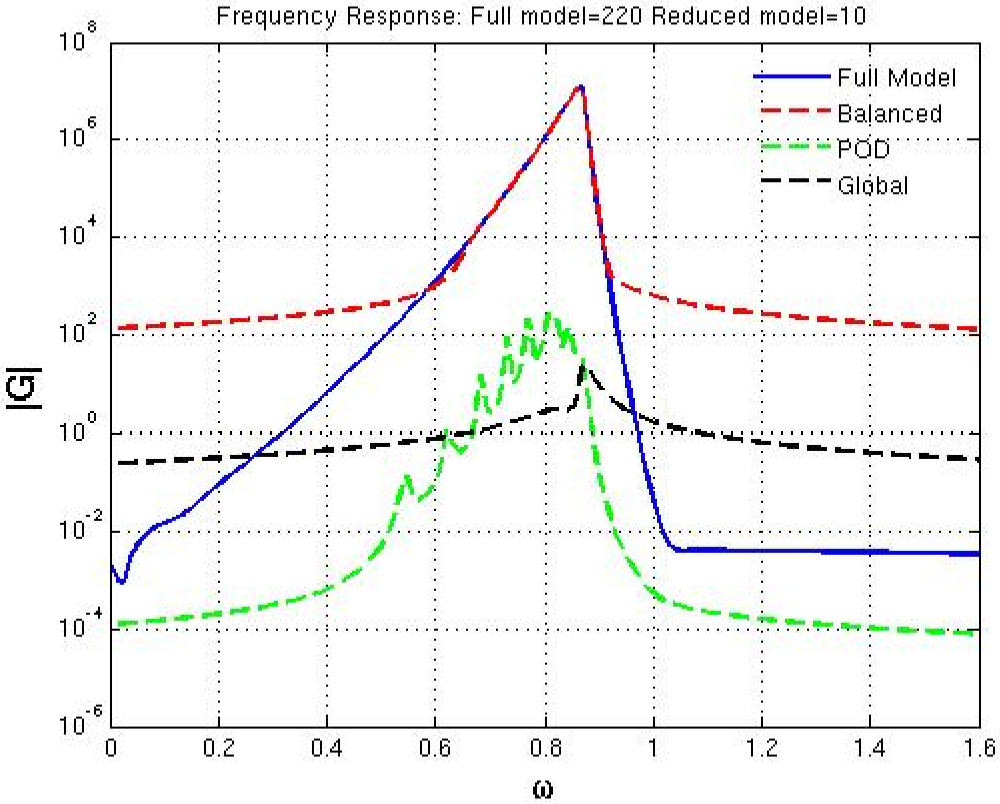
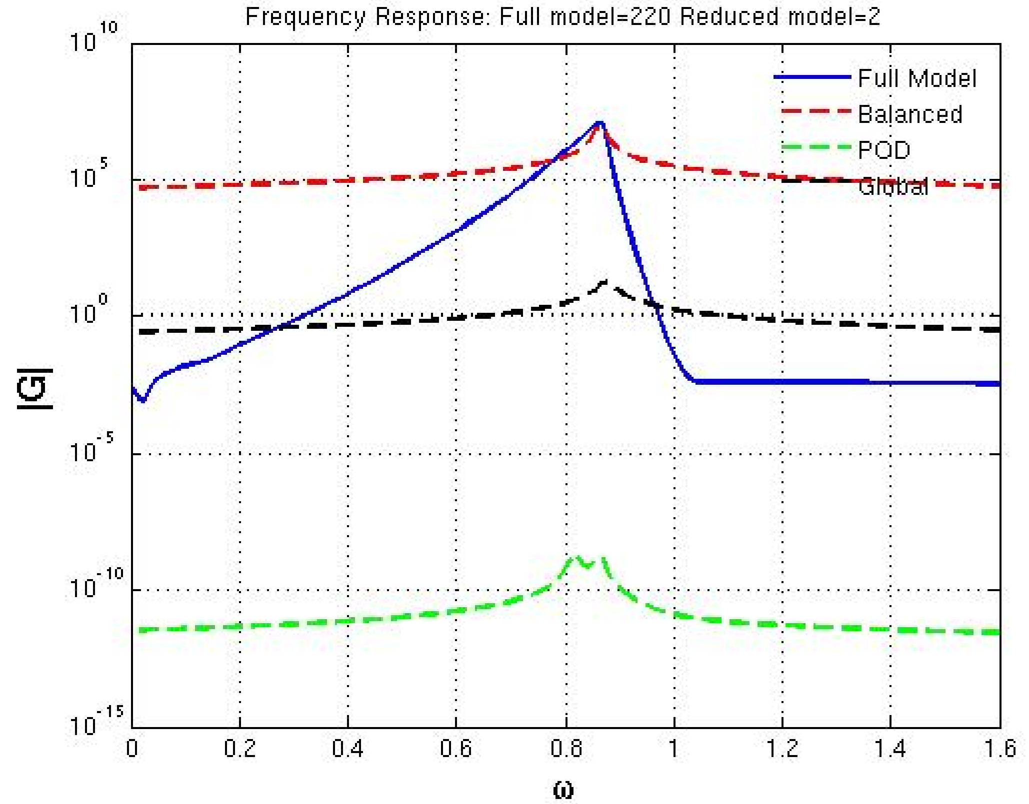
White noise -> Sensor



- B close to branch I and C close to branch II
- Full-order: n=220
- Reduced-order: m=30
- Balanced modes perform best

White noise -> Sensor



- B close to branch I and C close to branch II
- Full-order: n=220
- Reduced-order: m=2
- Balanced modes perform best again

LQG -Feedback control (1)

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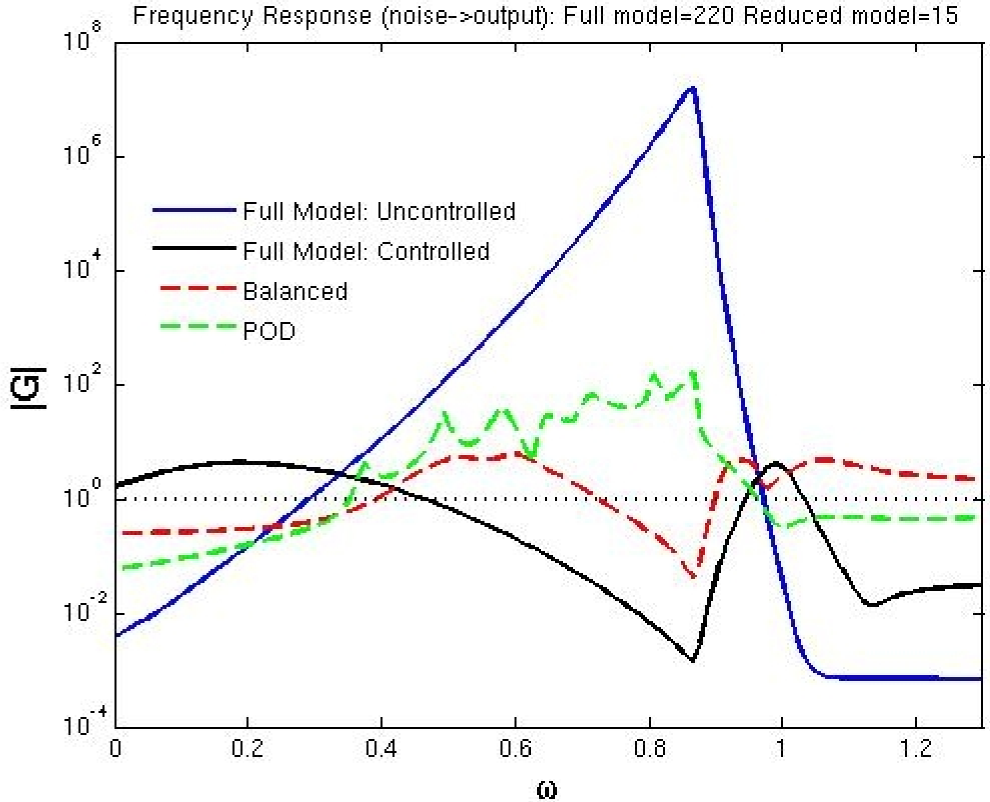
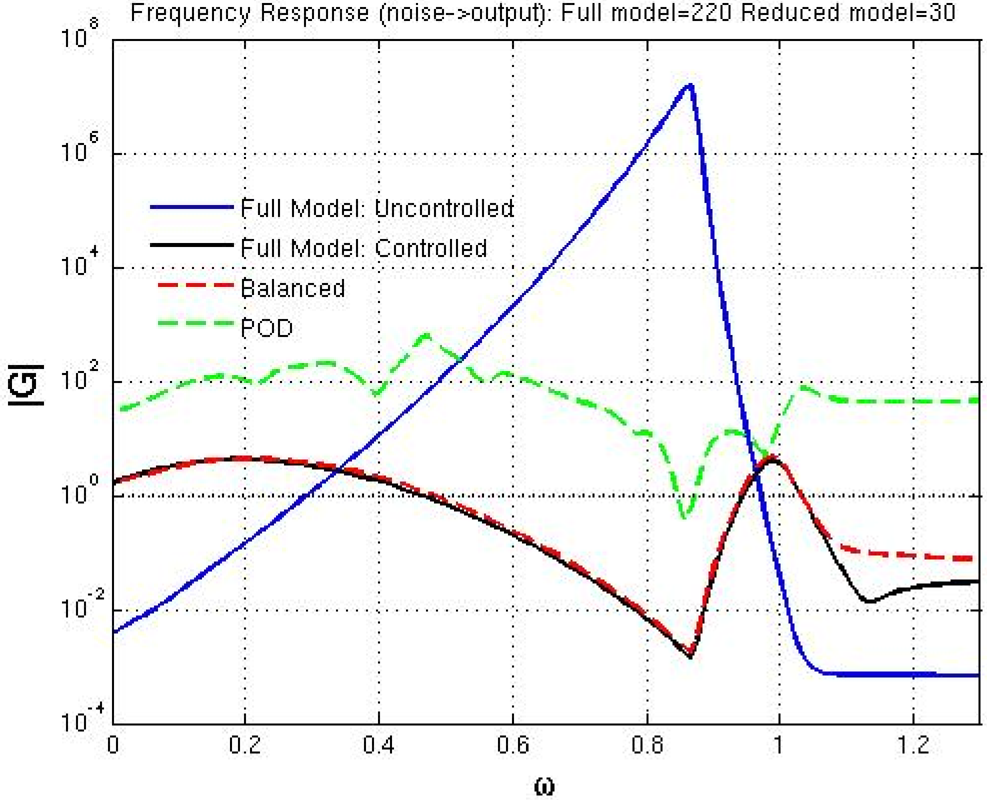


- Inputs & Outputs:
  - Disturbance: White noise
  - Actuator: Gaussian Volume forcing
  - Sensor: Gaussian function
- Full-order: n=220, Reduced-order: m=15

LQG -Feedback control (2)



- Inputs & Outputs:
  - Disturbance: White noise
  - Actuator: Gaussian Volume forcing
  - Sensor: Gaussian function
- Full-order: n=220, Reduced-order: m=30

Conclusions

- Balanced modes perform better than POD modes approximating the large-scale model for the examples studied here.

- Balanced truncation is extendable to large-scale systems, using the snapshot method. Cost similar to computing POD modes.

- These mode capture the relation between input signals and output signals, and are hence suitable for control design.

Extra slides

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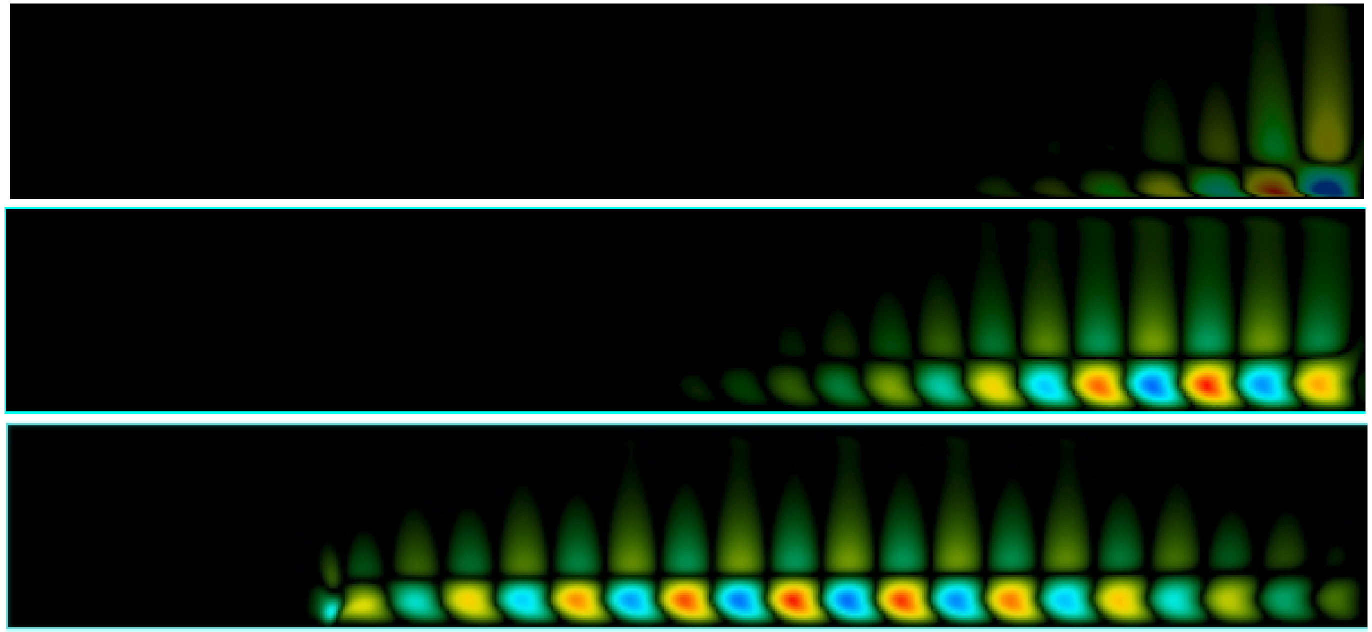
Blasius Modes



Matrix-free methods:

- Arnoldi method
- Snapshot method
- Snapshot method of direct and adjoint equations

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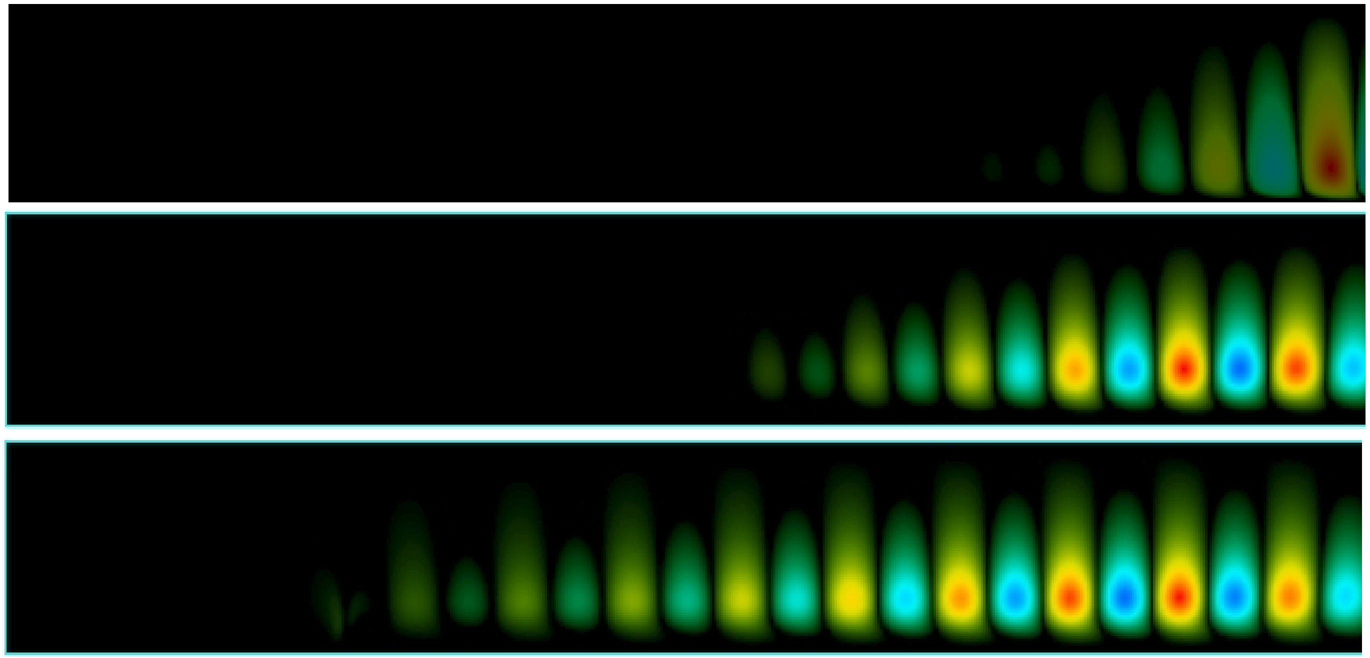
Blasius Modes



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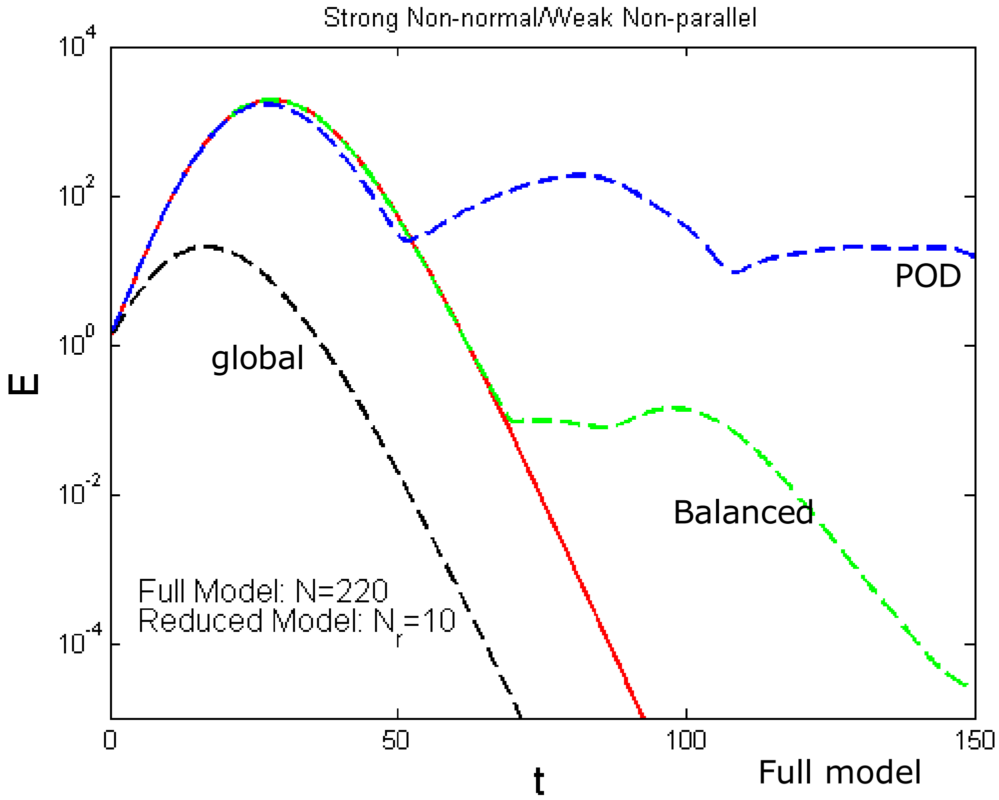
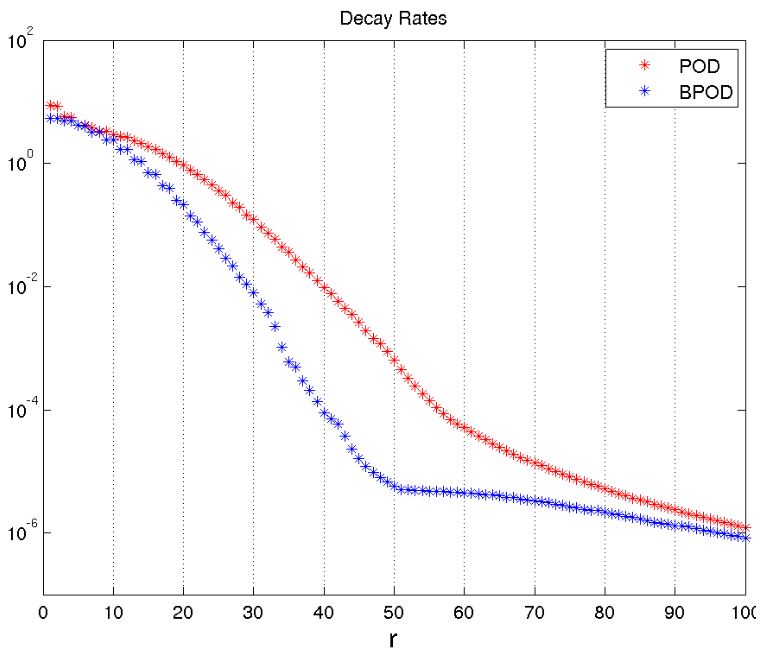
Optimal Growth

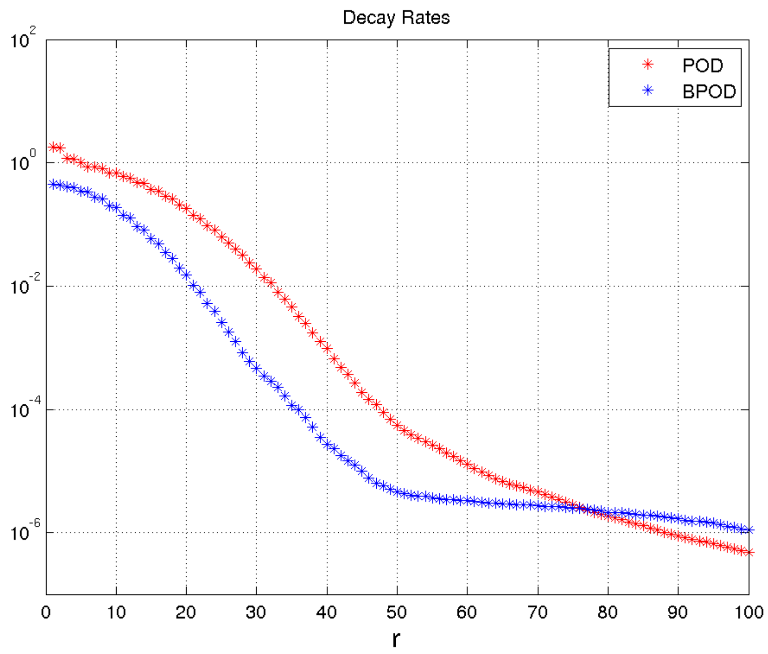


- B and C are at branch I and II respectively.
- Full-order: n=220.
- Reduced-order: m=10
- Balanced modes performs best

Decay rates for Blasius



u-component



v-component

Examples of model reduction for fluid systems

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- Control and optimal growth of globally unstable cavity-like flow using 2D global modes. Åkervik et al. (Jfm, 2007)
- Projection of actuators and sensors, frequency response and optimal growth using balanced modes on the 3D channel flow and 2D aerofoil. Rowley et al 2005-2007.
- POD modes (with shift mode) to describe flow around circular buildings. Noack et al (jfm, 2003).

Comparision of modes

- Spatial support.

of modes,

$$\frac{\text{abs}(V)}{\text{max}(V)} < 0.99$$