Stability analysis and control design of complex flows using timestepping techniques

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The aim of the presentation

- How to make use of Navier-Stokes solver for stability analysis and control design
- Modal and non-modal stability using DNS
- Input-output (I/O) formulation for model reduction and control
- The flat-plate boundary layer is considered
The flat-plate

- **Stability**: Behavior of small-amplitude perturbations

- **Control**: Reduce growth of small-amplitude perturbations
Global approach

- Linearized Navier-Stokes equations about a baseflow

\[
\frac{\partial u}{\partial t} + (\mathbf{U} \cdot \nabla) u + (u \cdot \nabla) \mathbf{U} = -\nabla p + \frac{1}{\text{Re}} \nabla^2 u,
\]

\[
\nabla \cdot u = 0,
\]

\[
u(x, 0) = u_0,
\]

- Initial value problem

\[
\frac{\partial u}{\partial t} = A u
\]

\[
u(0) = u_0,
\]

- Solution

\[
u(t) = e^{At} u_0
\]

- Investigate the properties of matrix exponential
Dimension of discretized system

<table>
<thead>
<tr>
<th></th>
<th>Base Flow</th>
<th>Inhomogeneous direction(s)</th>
<th>Dimension of ( u(t) )</th>
<th>Storage of ( A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ginzburg-Landau</td>
<td>( U(x) )</td>
<td>1D</td>
<td>( 10^2 )</td>
<td>1 MB</td>
</tr>
<tr>
<td>Blasius</td>
<td>( U(x,y) )</td>
<td>2D</td>
<td>( 10^5 )</td>
<td>25 GB</td>
</tr>
<tr>
<td>Jet in crossflow</td>
<td>( U(x,y,z) )</td>
<td>3D</td>
<td>( 10^7 )</td>
<td>500 TB</td>
</tr>
</tbody>
</table>

- Matrix \( A \) very large for spatially developing flows
- Use Navier-Stokes solver (DNS) or any CFD code to approximate the action of exponential matrix:

\[
u(\Delta t) = e^{A(\Delta t)} u_0\]

- Time-stepper technique: Never store matrices and use only velocity fields

\[ u_0 \xrightarrow{DNS} u(\Delta t) \]
Modal and non-modal stability

- Asymptotic behavior:

  \[ e^{At}u_j = \sigma_j u_j \]

  \[ |\sigma_1| > 1 \quad \text{globally unstable} \]
  \[ |\sigma_1| \leq 1 \quad \text{globally stable} \]

  - \( u_j \) global eigenmodes
  - Determine growth/decay as \( t \to \infty \)

- Short-time behavior:

  \[ e^{A^*t}e^{At}u_j = \sigma_j u_j \]

  \[ |\sigma_1| > 1 \quad \text{short-time growth} \]
  \[ |\sigma_1| \leq 1 \quad \text{monotonic decay} \]

  - \( u_j \) optimal disturbances
  - Determine growth/decay at fixed time \( t \)
Iterative eigenvalue methods

- Eigenvalue problem

\[ \mathcal{F}(\Delta t)u_j = \sigma_j u_j \quad (n \times n), \quad n > 10^5 \]

- Construct a small subspace from snapshots

\[ \mathcal{K} = \text{span}\{u_0, \mathcal{F}(\Delta t)u_0, \mathcal{F}(2\Delta t)u_0, \ldots, \mathcal{F}((m-1)\Delta t)u_0\} \]

- Solve small eigenvalue problem

  - Orthonormalize (e.g. Arnoldi) \( V = [V_1, \ldots, V_m] \)
  
  - Project operator \( \mathcal{F}(\Delta t) \approx VHVT \)
  
  - Solve small eigenvalue problem \( HS = S\Sigma \quad (m \times m), \quad m < 100 \)
Global eigenmodes

- The eigenvalue problem

\[ e^{At} u_j = \sigma_j u_j \]

- Subspace

\[ \mathcal{K} = \{ u_0, e^{A \Delta t} u_0, e^{A \Delta t}^2 u_0, \ldots, e^{A (m-1) \Delta t} u_0 \} \]

- Basis vector: snapshots of flow fields separated by constant time

\[ T_0 < \Delta t < T_{\text{Nyquist}} \]

- Eigenvalues of A recovered from

\[ \omega = \ln(\sigma) / \Delta t \]
Global spectrum

- Timestepper in red
- Globally stable
- Spatial support downstream
- Tollmien-Schlichting wavepackets
Optimal disturbances

- The eigenvalue problem
  \[ e^{A^* t} e^{A t} u_j = \sigma_j u_j \]

- Subspace
  \[ \mathcal{K} = \{ u_0, e^{A^* \Delta t} e^{A \Delta t} u_0, \ldots, e^{A^* (m-1) \Delta t} e^{A (m-1) \Delta t} u_0 \} \]

- Basis vectors: snapshots of adjoint flow fields separated by a fixed time

- Modes are orthogonal
Spectrum of optimal disturbances

- Time = 1800
- Eigenvalues come in pair
- Suboptimal have order of magnitude smaller energy
- Spatial support upstream
- Tilted upstream direction
Non-modal growth

- Repeat different times: $\Delta t$
- $10^4$ transient growth
Inputs and outputs

- $B_1$ is disturbance – volume forcing
- $u_w$ is actuator - blowing/suction at wall
- $C_1$ and $C_2$ are sensors - shear stress at wall

$$\dot{u} = Au + B_1w,$$
$$u(x, 0, t) = u_w \phi(t),$$
$$z(t) = C_1 u,$$
$$y(t) = C_2 u.$$
Lifting procedure

• Wall actuation is inhomogenous boundary condition

\[ \dot{u} = Au \]
\[ u(x, 0, t) = u_w \phi(t), \]

• Split solution into:

\[ u = u_h + u_p \]

• Find any particular solution, e.g. steady solution: \( u_p = -B_2 \)

\[ 0 = Au_p, \]
\[ u_p(x, 0, t) = u_w \phi(t), \]

• “Lift” boundary term to volume forcing

\[ \dot{u}_h = Au_h + B_2 \psi \]
\[ \psi = \dot{\phi} \]
H2 – Feedback controller

- Add measurement noise ($\alpha$) and control penalty ($l$)

- Objective function
  \[ \|z\|^2 = \int_0^t u^T C_1^T C_1 u + l^2 \psi^2 \, dt \]

- Find control signal $\psi(t)$ based on the measurements $y(t)$ such that the influence of external disturbances $w(t)$ and $g(t)$ on the output $z(t)$ is minimized.
Standard State-space formulation

- State-space with control penalty and measurement noise

\[
\dot{\mathbf{u}} = A\mathbf{u} + B_1 \mathbf{w} + B_2 \psi, \\
\mathbf{z}(t) = C_1 \mathbf{u} + l\psi \\
\mathbf{y}(t) = C_2 \mathbf{u} + \alpha g
\]

- Three inputs, \( w = (w, g, \psi)^T \): \( B = (B_1, 0, B_2) \)

- Two outputs, \( z = (z, y)^T \) \( C = (C_1, C_2)^T \)

- Feed-through

\[
D = \begin{pmatrix} 0 & 0 & l \\ 0 & \alpha & 0 \end{pmatrix}
\]

- Standard state-space formulation

\[
\dot{\mathbf{u}} = A\mathbf{u} + B\mathbf{w}, \\
\mathbf{z}(t) = C\mathbf{u} + D\mathbf{w}
\]
Model reduction

• Approximate the large system

\[
\begin{align*}
\dot{u} &= Au + Bw, \\
z &= Cu + Dw
\end{align*}
\]

with a small system

\[
\begin{align*}
\dot{\hat{u}} &= A_r \hat{u} + B_r w, \\
\hat{z} &= C_r \hat{u} + D w
\end{align*}
\]

so that the I/O behavior is preserved:

\[
\sup_w \frac{\|z - \hat{z}\|_2}{\|w\|_2} = \|G - G_r\|_\infty \leq \epsilon(r)
\]

• One systematic approach is balanced truncation (Moore 1981)

\[
\sigma_{r+1} \leq \|G - G_r\|_\infty \leq 2 \sum_{i=r+1}^{n} \sigma_i
\]
Controllability

- Flow states most easily excited by input
- Diagonalize the correlation matrix of the flow

\[ P = \int_0^\infty e^{A\tau} BB^* e^{A^*\tau} d\tau \]

- POD modes
Observability

- Flow states that will most easily excite output
- Diagonalize the correlation matrix of adjoint flow
- Adjoint POD modes

\[ Q = \int_0^\infty e^{A^* \tau} C^* C e^{A \tau} \, d\tau \]
Balancing

• Controllability

\[ P = \int_0^\infty e^{A \tau} B B^* e^{A^* \tau} \, d\tau \]

• Observability

\[ Q = \int_0^\infty e^{A^* \tau} C^* C e^{A \tau} \, d\tau \]

• Balanced modes

Diagonalize both controllability and observability Gramian

\[ PQ u_j = \sigma_j^2 u_j \quad (n \times n), \quad n > 10^5 \]
The snapshot method

Snapshots of direct simulation

\[ X = \{ B_1, \ldots, e^{A\Delta t}B_1, B_2, \ldots, e^{A\Delta t}B_p \} \]

Snapshots of adjoint simulation

\[ Y = \{ C_1^*, \ldots, C_r^*, e^{A^*\Delta t}C_2^*, \ldots, e^{A^*\Delta t}C_r^* \} \]
The snapshot method

- Singular value decomposition of size: \((pm \times rm)\)
  \[ Y^T X = U \Sigma V^T \]

- Balanced modes
  \[ T = XV \]

- Adjoint balanced modes
  \[ S = YU \]
Performance of reduced system

DNS: $n=10^5$

ROM: $m=50$
Model reduction error

- Frequency response
  All inputs to all outputs
  Order 2 captures the I/O behavior
  Order 80 captures all frequencies

- Theoretical error bounds

\[ \sigma_{r+1} \leq \| G - G_r \|_\infty \leq 2 \sum_{i=r+1}^{n} \sigma_i \]
Performance of controller

- Disturbance input white-noise
- Energy minimized at sensor 2
Control of disturbance

No control: $10^4$ growth

Linear control: $10^2$ growth
3D disturbances in flat-plate

- Disturbances:
  - Free-stream turbulence
  - Streaks

- Rows of inputs and outputs
  - Size and shape
  - Spacing

- Apply linear controller to nonlinear flow

- Robustness

- KTH/MTL windtunnel
The snapshot method

- If number of inputs (p) and outputs (r) are small, e.g. <10-50

- Extension: Output projection (Rowley, 2005) if either number of inputs or outputs is large

\[ X = \{B_1, \ldots, e^{A\Delta t} B_1, e^{A\Delta t} B_2, \ldots, e^{A\Delta t} B_p\} \]

\[ Y = \{C_1^*, \ldots, e^{A^*\Delta t} C_1^*, e^{A^*\Delta t} C_2^*, \ldots, e^{A^*\Delta t} C_r^*\} \]
3D balanced modes

First Balanced mode

First adjoint Balanced mode
Balanced modes by iteratively techniques

- The eigenvalue problem
  \[ PQu_j = \sigma_j u_j \]

- Subspace
  \[ \mathcal{K} = \{ u_0, \int_0^{\Delta t} e^{A\tau} BB^* e^{A^*\tau} d\tau \int_0^{\Delta t} e^{A^*\tau} C^* C e^{AT\tau} d\tau u_0, \ldots \} \]

- Basis vector snapshots of flow fields separated by constant time

- Backward to compute adjoint modes
Conclusions

- Existing CFD codes can be used for modern stability analysis and control design
- Computational cost of the same order as numerical simulations
- Swept wing, Separated flows, Flows over steps and cavities, Flows in ducts and corners, Wake-vortex flows and bluff bodies
Jet in crossflow

Counter rotating vortex pair

Horseshoe vortices

Shear layer vortices

Wake region

\[ R = \frac{U}{V} = 3 \]
\[ \text{Re} = \frac{U \delta_0}{\nu} = 165 \]
Direct Numerical Simulation

DNS simulation

Steady-State (SFD)
Fully 3D global eigenmodes
Nekton 5000

- Spectral element code by Paul Fischer, Argonne National Laboratory
- 80,000 lines of f77 (some C)
- Structured grid – rectangular elements
- Curved geometries
- Massively parallel – 32,000 cores
Extra slides