Stability analysis and control design
of complex flows using timestepping techniques

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The aim of the presentation

- How to make use of Navier-Stokes solver for stability analysis and control design
- Modal and non-modal stability using DNS
- Input-output (I/O) formulation for model reduction and control
- The flat-plate boundary layer is considered

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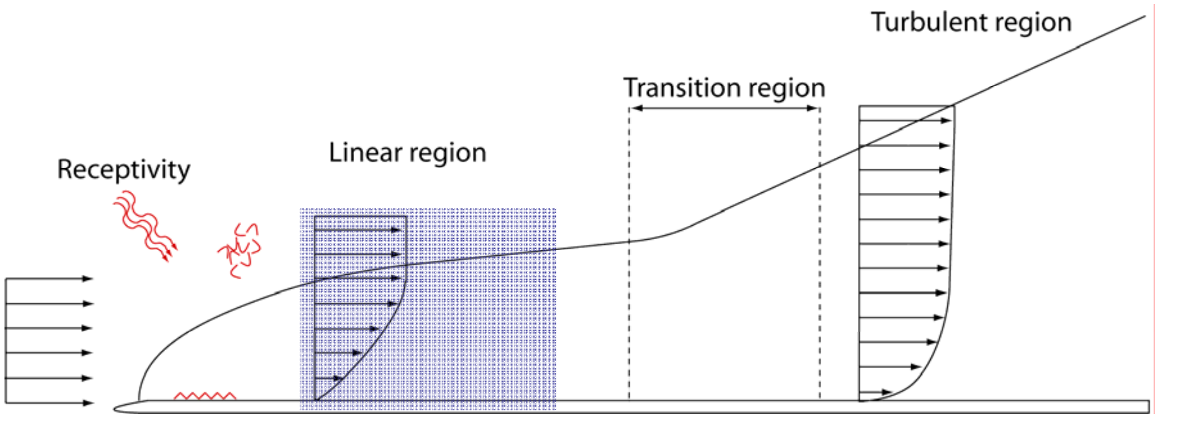
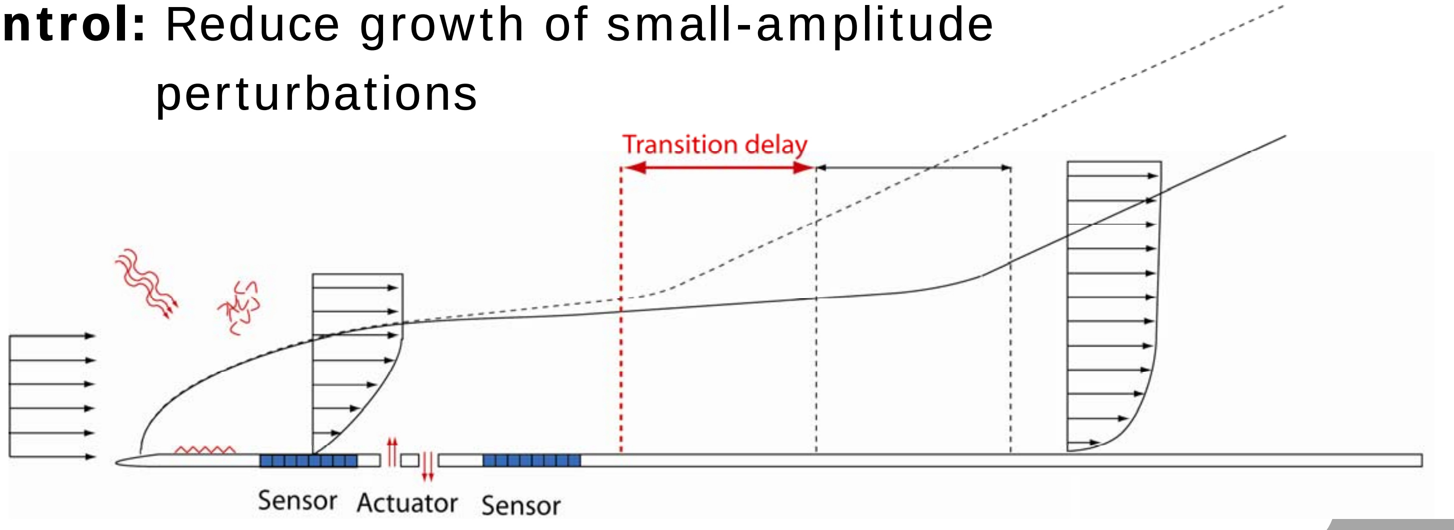
The flat-plate

- **Stability:** Behavior of small-amplitude perturbations



- **Control:** Reduce growth of small-amplitude perturbations



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Global approach

- Linearized Navier-Stokes equations about a baseflow

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + (\mathbf{U}\cdot\nabla)\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\mathbf{U} &= -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2 \boldsymbol{u}, \\ \nabla\cdot\boldsymbol{u} &= 0, \\ \boldsymbol{u}(\mathbf{x},0) &= \boldsymbol{u}_0, \end{aligned}$$

- Initial value problem

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} &= A\boldsymbol{u} \\ \boldsymbol{u}(0) &= \boldsymbol{u}_0, \end{aligned}$$

- Solution

$$\boldsymbol{u}(t) = e^{At}\boldsymbol{u}_0$$

- Investigate the properties of matrix exponential

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Dimension of discretized system

	Base Flow	Inhomogeneous direction(s)	Dimension of $\boldsymbol{u}(t)$	Storage of A
Ginzburg-Landau	$U(x)$	1D	10^2	1 MB
Blasius	$U(x,y)$	2D	10^5	25 GB
Jet in crossflow	$U(x,y,z)$	3D	10^7	500 TB

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- Matrix A very large for spatially developing flows
- Use Navier-Stokes solver (DNS) or any CFD code to approximate the action of exponential matrix:

$$\mathbf{u}(\Delta t) = e^{A(\Delta t)}\mathbf{u}_0$$

- Time-stepper technique: Never store matrices and use only velocity fields

Modal and non-modal stabilility

- Asymptotic behavior:

$$e^{At}\mathbf{u}_j = \sigma_j \mathbf{u}_j$$

$|\sigma_1| > 1$ globally unstable

$|\sigma_1| \leq 1$ globally stable

  - u_j global eigenmodes
  - Determine growth/decay as t$\rightarrow \infty$

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- Short-time behavior:

$$e^{A^* t} e^{At}\mathbf{u}_j = \sigma_j \mathbf{u}_j$$

$|\sigma_1| > 1$ short-time growth

$|\sigma_1| \leq 1$ monotonic decay

  - u_j optimal disturbances
  - Determine growth/decay at fixed time t

Iterative eigenvalue methods

- Eigenvalue problem

$$\mathcal{F}(\Delta t)\mathbf{u}_j = \sigma_j \mathbf{u}_j \qquad (n \times n), \qquad n > 10^5$$

- Construct a small subspace from snapshots

$$\mathcal{K} \quad = \quad \mathrm{span}\{\mathbf{u}_0, \mathcal{F}(\Delta t)\mathbf{u}_0, \mathcal{F}(2\Delta t)\mathbf{u}_0, \ldots, \mathcal{F}((m-1)\Delta t)\mathbf{u}_0\}$$

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- Solve small eigenvalue problem
  - Orthonormalize (e.g. Arnoldi) $\mathbf{V} = [V_1, \ldots, V_m]$
  - Project operator $\mathcal{F}(\Delta t) \approx \mathbf{V}\mathbf{H}\mathbf{V}^T$
  - Solve small eigenvalue problem $\mathbf{H}\mathbf{S} = \mathbf{S}\mathbf{\Sigma} \qquad (m \times m), \quad m < 100$

Global eigenmodes

- The eigenvalue problem

$$e^{At}\mathbf{u}_j = \sigma_j \mathbf{u}_j$$

- Subspace

$$\mathcal{K} = \{\mathbf{u}_0, e^{A\Delta t}\mathbf{u}_0, e^{A2\Delta t}\mathbf{u}_0, \ldots, e^{A(m-1)\Delta t}\mathbf{u}_0\}$$

- Basis vector: snapshots of flow fields separated by constant time

$$T_\text{O} < \Delta t < T_\text{Nyquist}$$

- Eigenvalues of A recovered from

$$\omega = \ln(\sigma)/\Delta t$$

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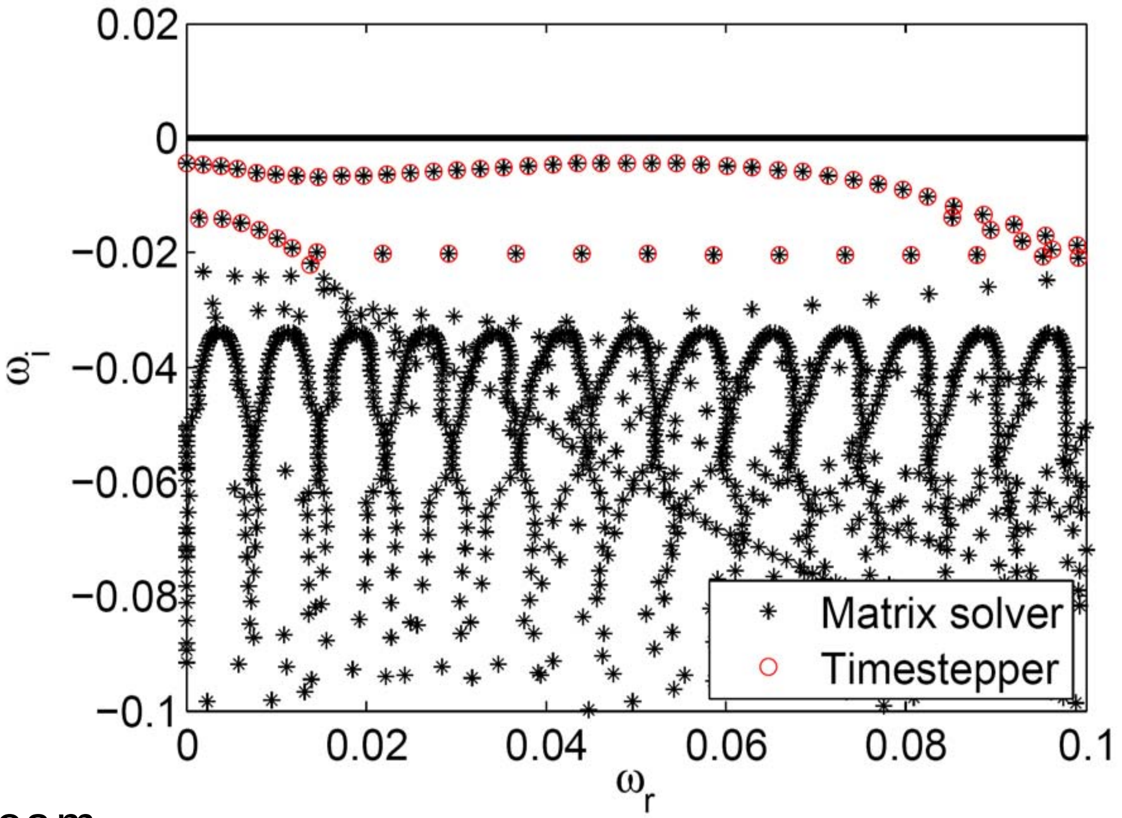
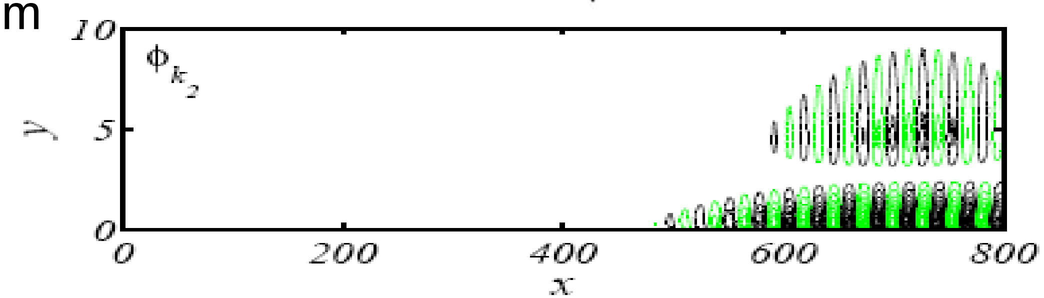
Global spectrum

- Timestepper in red
- Globally stable



- Spatial support downstream
- Tollmien-Schlichting wavepackets



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Optimal disturbances

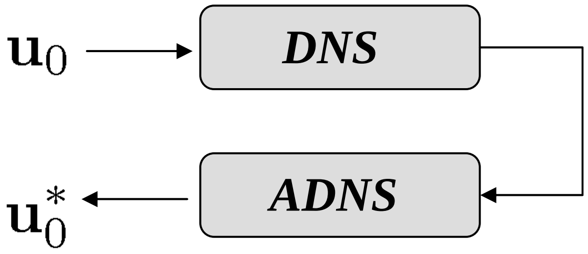
- The eigenvalue problem

$$e^{A^* t} e^{At} \mathbf{u}_j = \sigma_j \mathbf{u}_j$$

- Subspace

$$\mathcal{K} = \{\mathbf{u}_0, e^{A^* \Delta t} e^{A \Delta t} \mathbf{u}_0, \ldots, e^{A^* (m-1) \Delta t} e^{A (m-1) \Delta t} \mathbf{u}_0\}$$

- Basis vectors: snapshots of adjoint flow fields separated by a fixed time



- Modes are orthogonal

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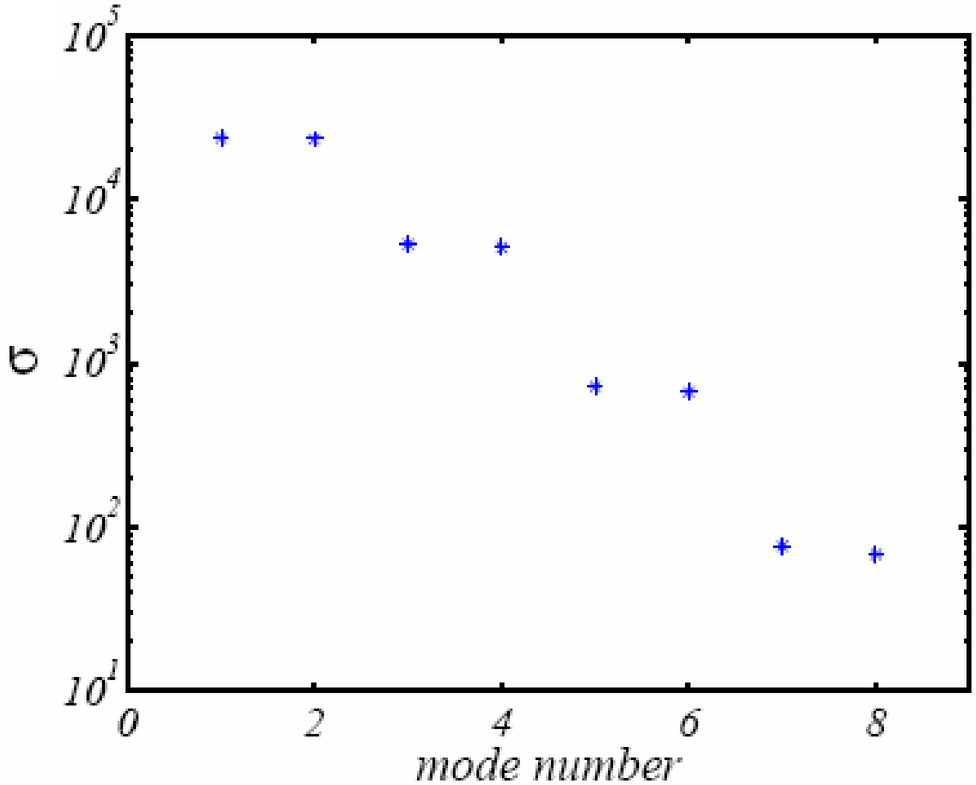
Spectrum of optimal disturbances

- Time = 1800
- Eigenvalues come in pair
- Suboptimal have order of magnitude smaller energy



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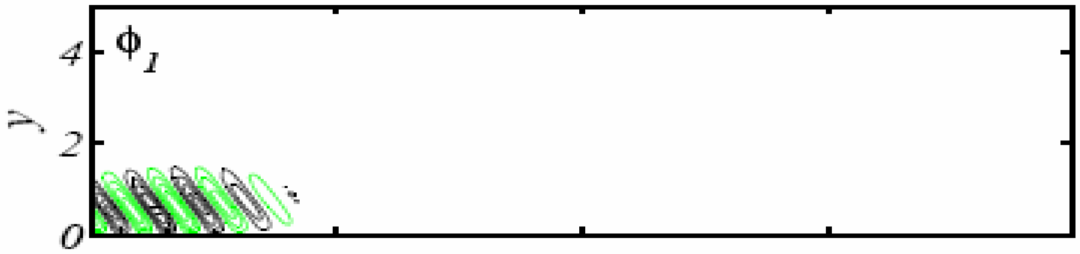
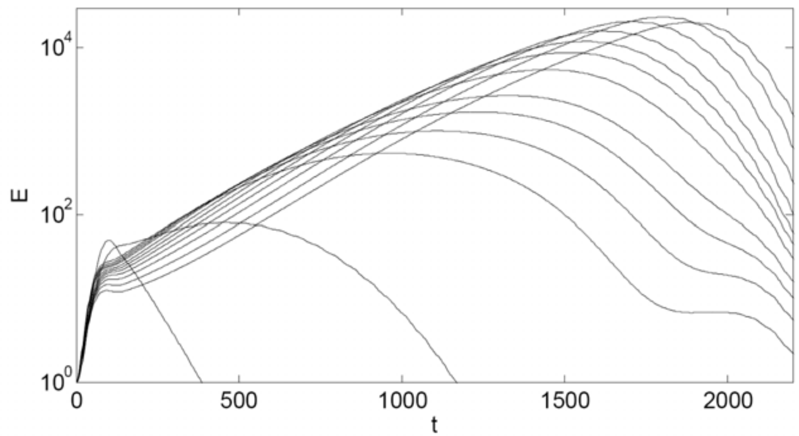
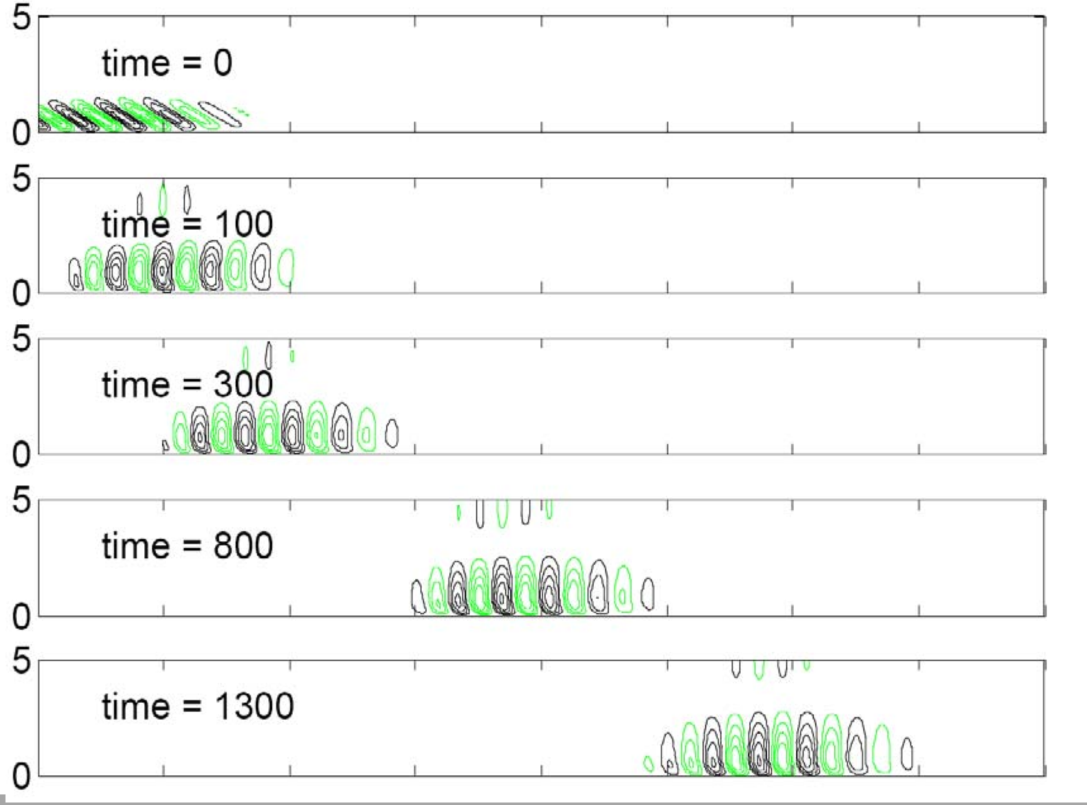
- Spatial support upstream
- Tilted upstream direction



Non-modal growth

- Repeat different times: Δ t
- 10^4 transient growth





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Inputs and outputs

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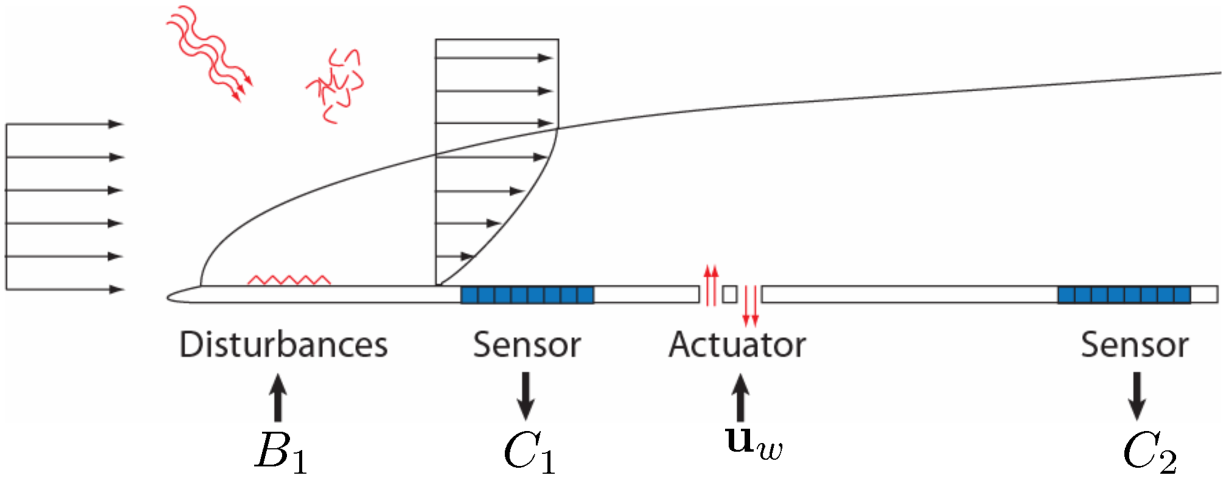

$$
\begin{aligned}
\dot{\mathbf{u}} &= A\mathbf{u} + B_1 w, \\
\mathbf{u}(x,0,t) &= \mathbf{u}_w \phi(t), \\
z(t) &= C_1 \mathbf{u} \\
y(t) &= C_2 \mathbf{u}.
\end{aligned}
$$

- B_1 is disturbance – volume forcing
- u_w is actuator - blowing/suction at wall
- C_1 and C_2 are sensors - shear stress at wall

Lifting procedure

- Wall actuation is inhomogenous boundary condition

$$\begin{aligned} \dot{\mathbf{u}} &= A\mathbf{u} \\ \mathbf{u}(x,0,t) &= \mathbf{u}_w \phi(t), \end{aligned}$$

- Split solution into:

$$\mathbf{u} = \mathbf{u}_h + \mathbf{u}_p$$

- Find any particular solution, e.g. steady solution: $\mathbf{u}_p = -B_2$

$$\begin{aligned} 0 &= A\mathbf{u}_p, \\ \mathbf{u}_p(x,0,t) &= \mathbf{u}_w \phi(t), \end{aligned}$$

- ”Lift” boundary term to volume forcing

$$\begin{aligned} \dot{\mathbf{u}}_h &= A\mathbf{u}_h + B_2\psi \\ \psi &= \dot{\phi} \end{aligned}$$

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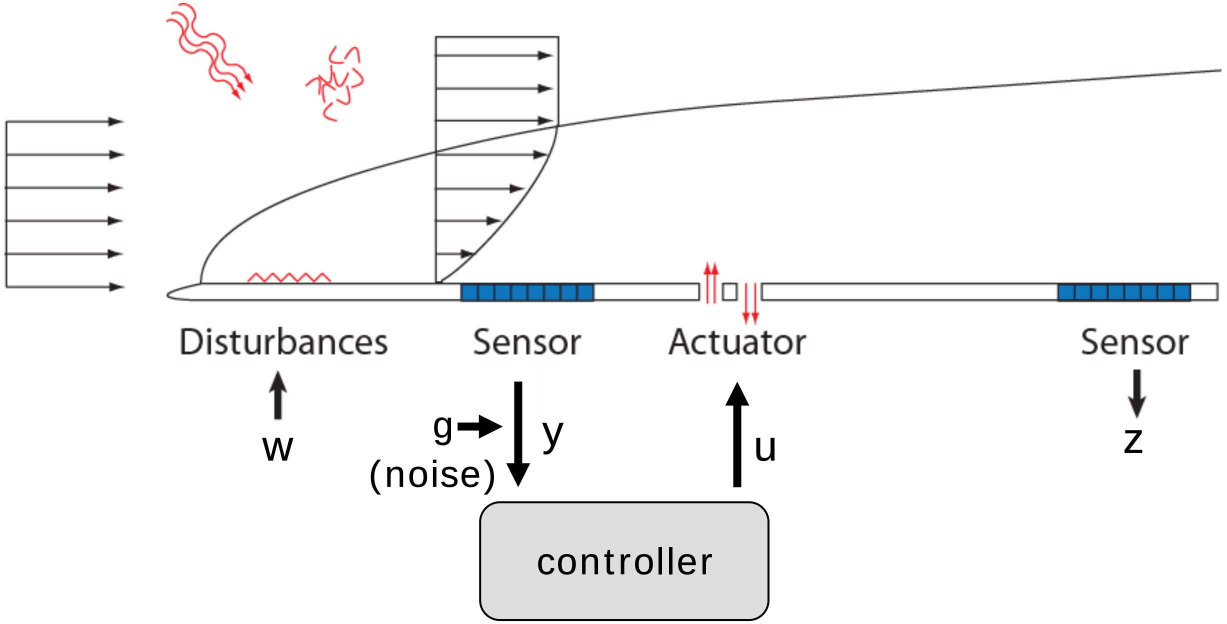
H2 – Feedback controller



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- Add measurement noise (α) and control penality (l)
- Objective function $\|z\|^2 = \int_0^t \mathbf{u}^T C_1^T C_1 \mathbf{u} + l^2 \psi^2 \, dt$
- *Find control signal ψ(t) based on the measurements y(t) such that the influence of external disturbances w(t) and g(t) on the output z(t) is minimized.*

Standard State-space formulation

- State-space with control penalty and measurement noise

$$\begin{aligned} \dot{\mathbf{u}} &= A\mathbf{u} + B_1 w + B_2 \psi, \\ z(t) &= C_1 \mathbf{u} + l\psi \\ y(t) &= C_2 \mathbf{u} + \alpha g \end{aligned}$$

- Three inputs, w=$(w, g, \psi)^T$: $B = (B_1, 0, B_2)$
- Two outputs, z=$(z, y)^T$ $C = (C_1, C_2)^T$
- Feed-trough $D = \begin{pmatrix} 0 & 0 & l \\ 0 & \alpha & 0 \end{pmatrix}$
- Standard state-space formulation

$$\begin{aligned} \dot{\mathbf{u}} &= A\mathbf{u} + B\mathrm{w}, \\ \mathrm{z}(t) &= C\mathbf{u} + D\mathrm{w} \end{aligned}$$

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Model reduction

- Approximate the large system

$$\begin{aligned} \dot{\mathbf{u}} &= A\mathbf{u} + B\mathrm{w}, \\ \mathrm{z} &= C\mathbf{u} + D\mathrm{w} \end{aligned} \qquad n > 10^5$$

with a small system

$$\begin{aligned} \dot{\hat{u}} &= A_r\hat{u} + B_r\mathrm{w}, \\ \hat{\mathrm{z}} &= C_r\hat{u} + D\mathrm{w} \end{aligned} \qquad r < 100$$

so that the I/O behavior is preserved:

$$\sup_{\mathrm{w}} \frac{\|\mathrm{z} - \hat{\mathrm{z}}\|_2}{\|\mathrm{w}\|_2} = \|G - G_r\|_\infty \leq \epsilon(r)$$

- One systematic approach is *balanced truncation* *(Moore 1981)*

$$\sigma_{r+1} \leq \|G - G_r\|_\infty \leq 2 \sum_{i=r+1}^{n} \sigma_i$$

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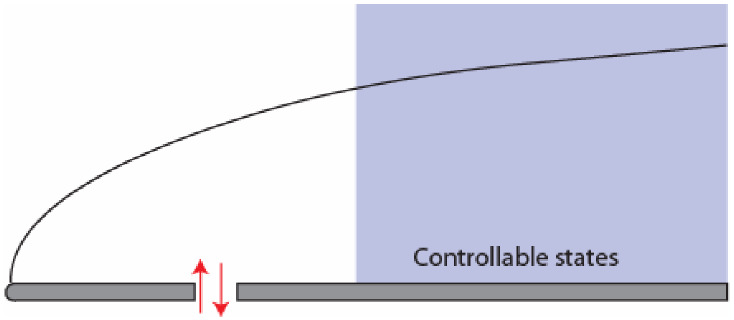
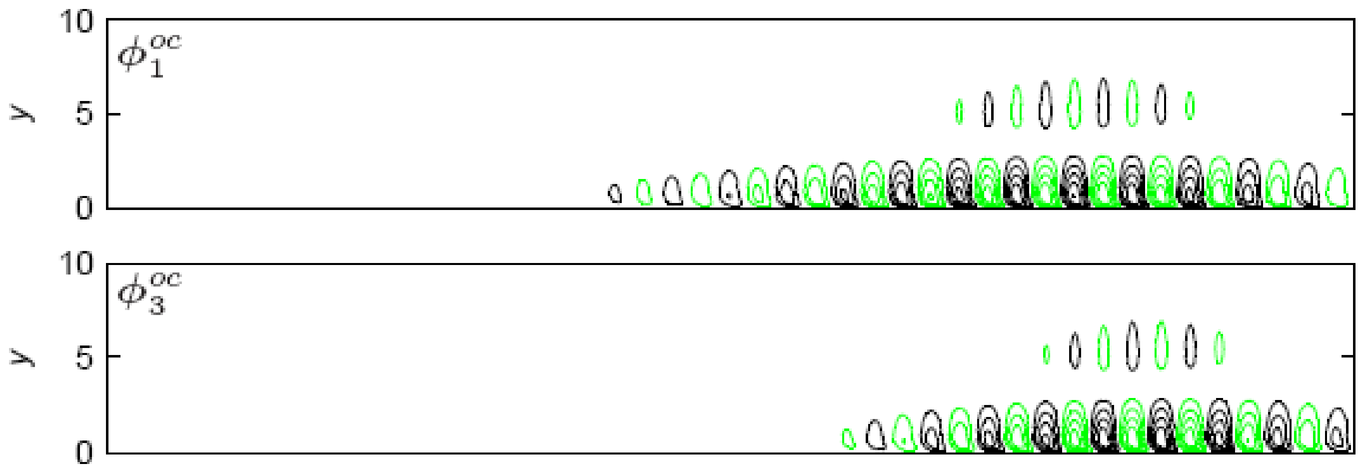
Controllability

- Flow states most easily excited by input
- Diagonalize the correlation matrix of the flow

$$P = \int_0^\infty e^{A\tau} B B^* e^{A^*\tau} \mathrm{d}\tau$$



- POD modes



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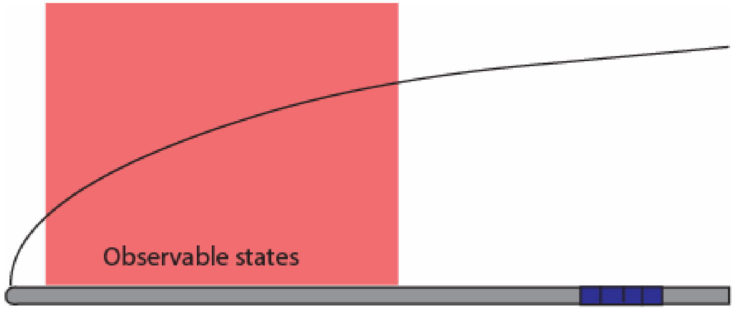
Observability

- Flow states that will most easily excite output
- Diagonalize the correlation matrix of adjoint flow

$$Q = \int_0^\infty e^{A^*\tau} C^* C e^{A\tau} \mathrm{d}\tau$$



- Adjoint POD modes

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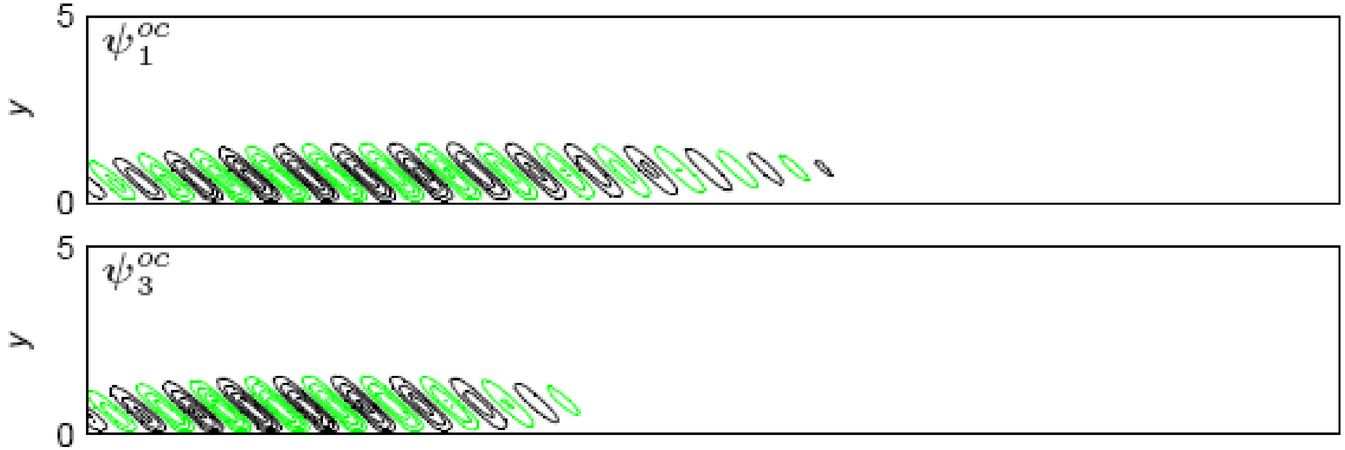
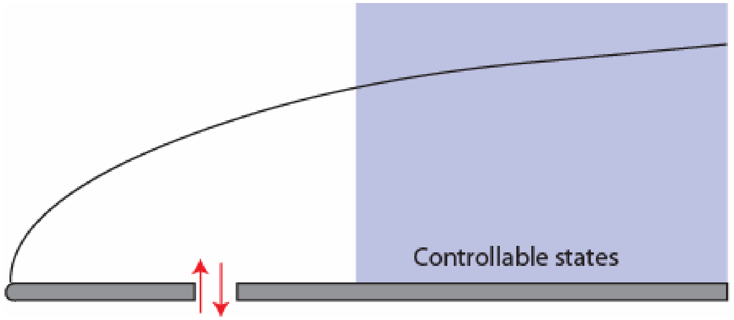


Balancing

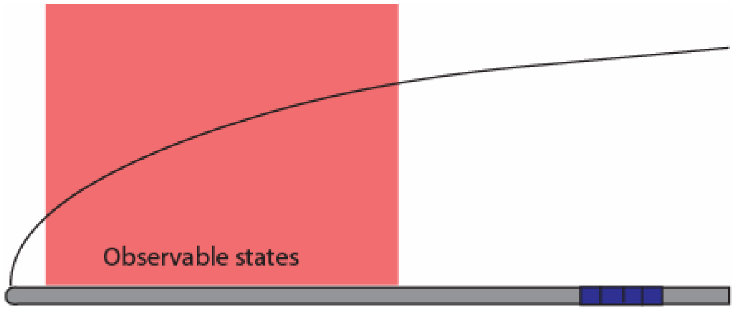
- Controllability

$$P = \int_0^\infty e^{A\tau} B B^* e^{A^*\tau} \mathrm{d}\tau$$



- Observability

$$Q = \int_0^\infty e^{A^*\tau} C^* C e^{A\tau} \mathrm{d}\tau$$



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- Balanced modes

  Diagonalize both controllability and observability Gramian

$$PQ\mathbf{u}_j = \sigma_j^2 \mathbf{u}_j \qquad (n \times n), \qquad n > 10^5$$

The snapshot method

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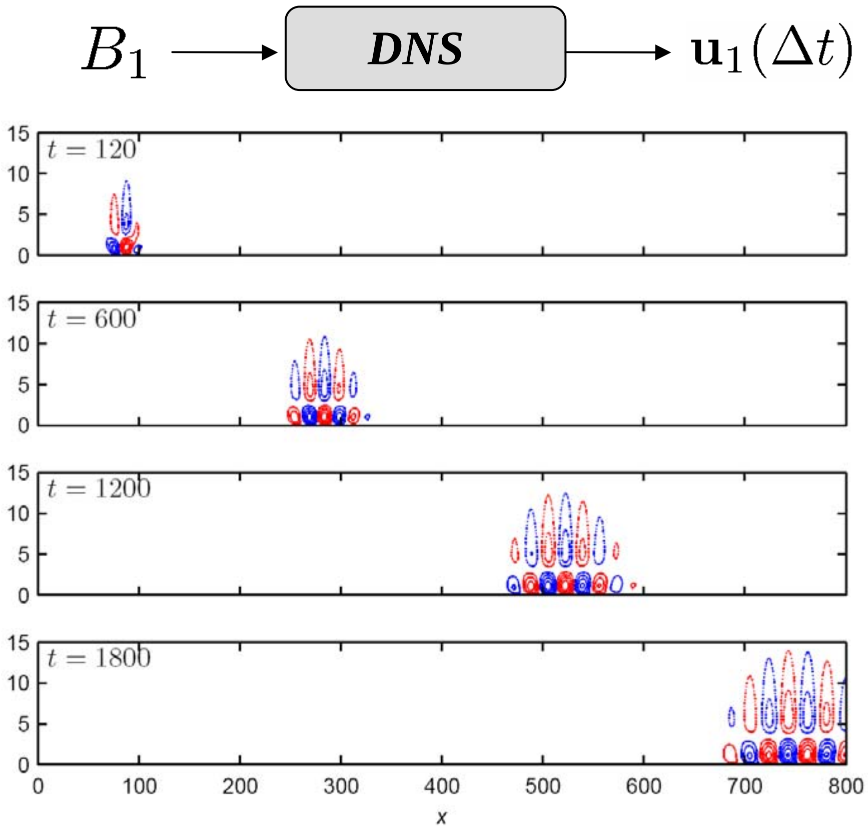
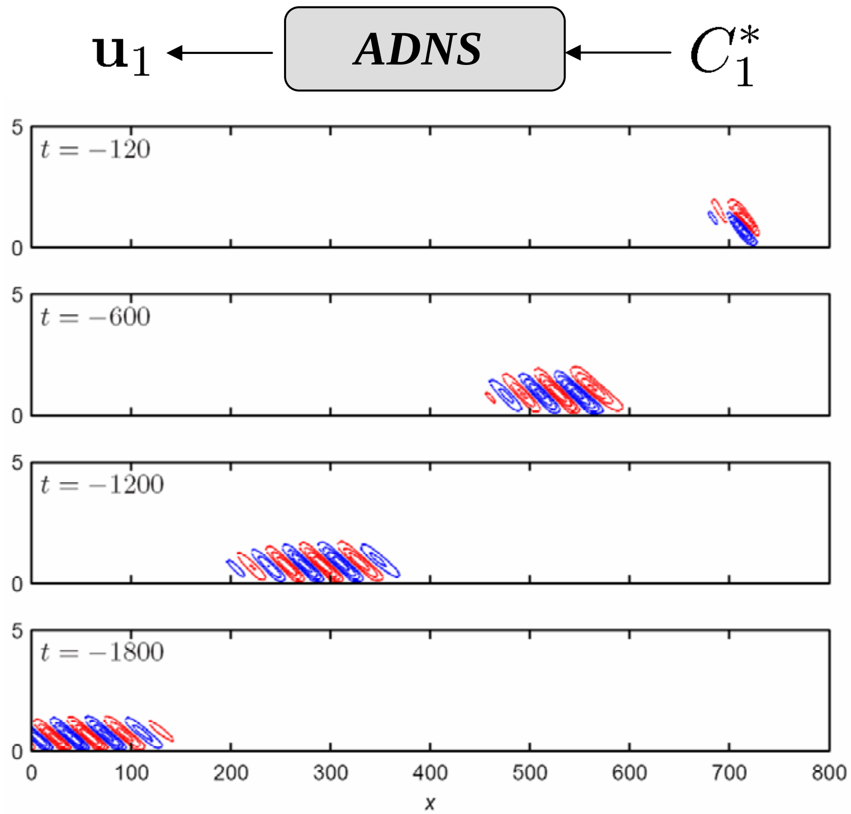
Snaphots of direct simulation

$$\mathbf{X} = \{B_1, \ldots, e^{A\Delta t}B_1, B_2, \ldots, e^{A\Delta t}B_p\}$$



Snapshots of adjoint simulation

$$\mathbf{Y} = \{C_1^*, \ldots, C_1^*, e^{A^*\Delta t}C_2^*, \ldots, e^{A^*\Delta t}C_r^*\}$$



The snapshot metod

- Singular value decomposition of size: (pm x rm)

$$\mathbf{Y}^T\mathbf{X} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$$

- Balanced modes

$$\mathbf{T} = \mathbf{X}\mathbf{V}$$

Adjoint balanced modes

$$\mathbf{S} = \mathbf{Y}\mathbf{U}$$

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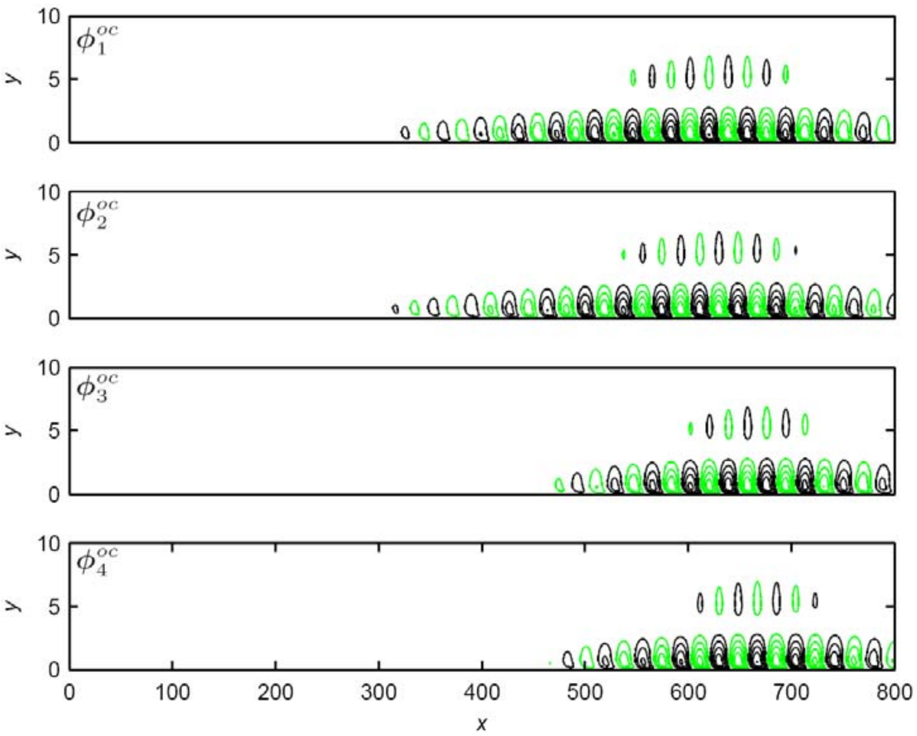
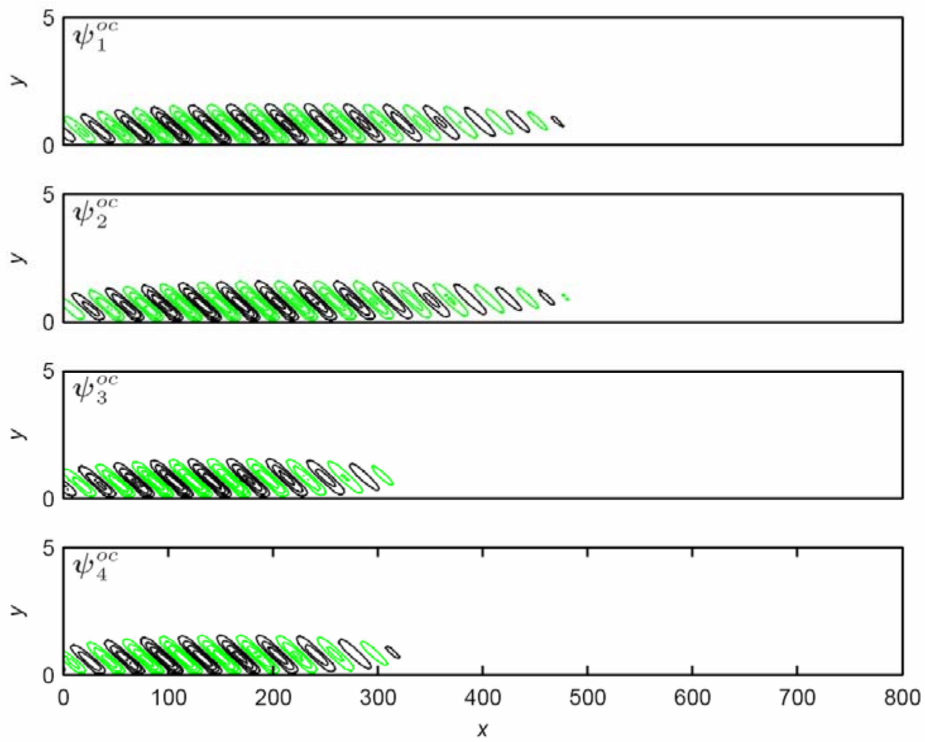
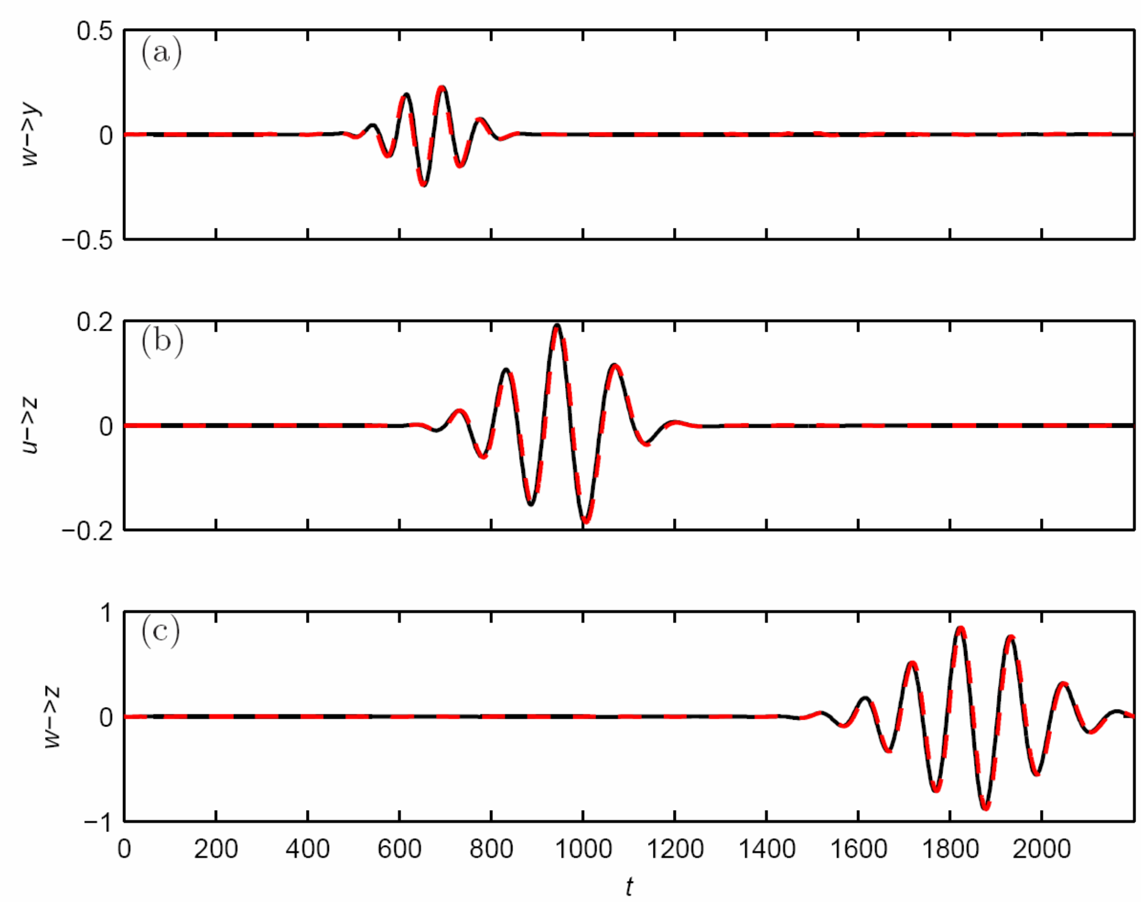




Performance of reduced system



Disturbance → Sensor

Actuator → Objective

Disturbance → Objective

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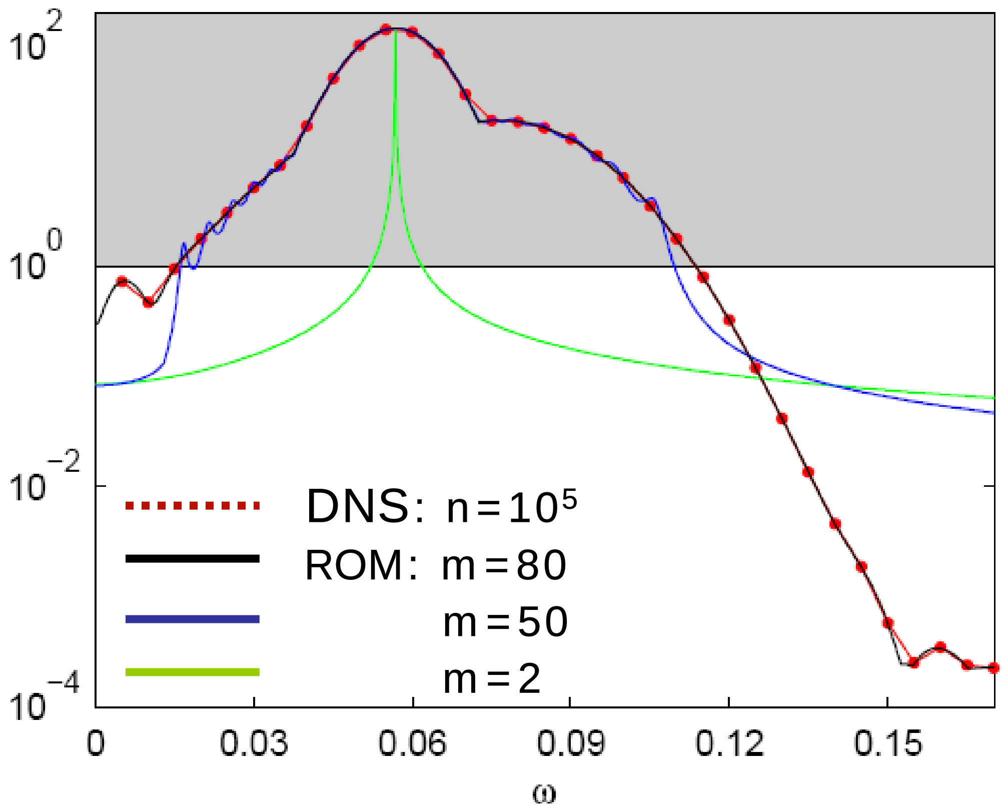
DNS: $n=10^5$

ROM: $m=50$

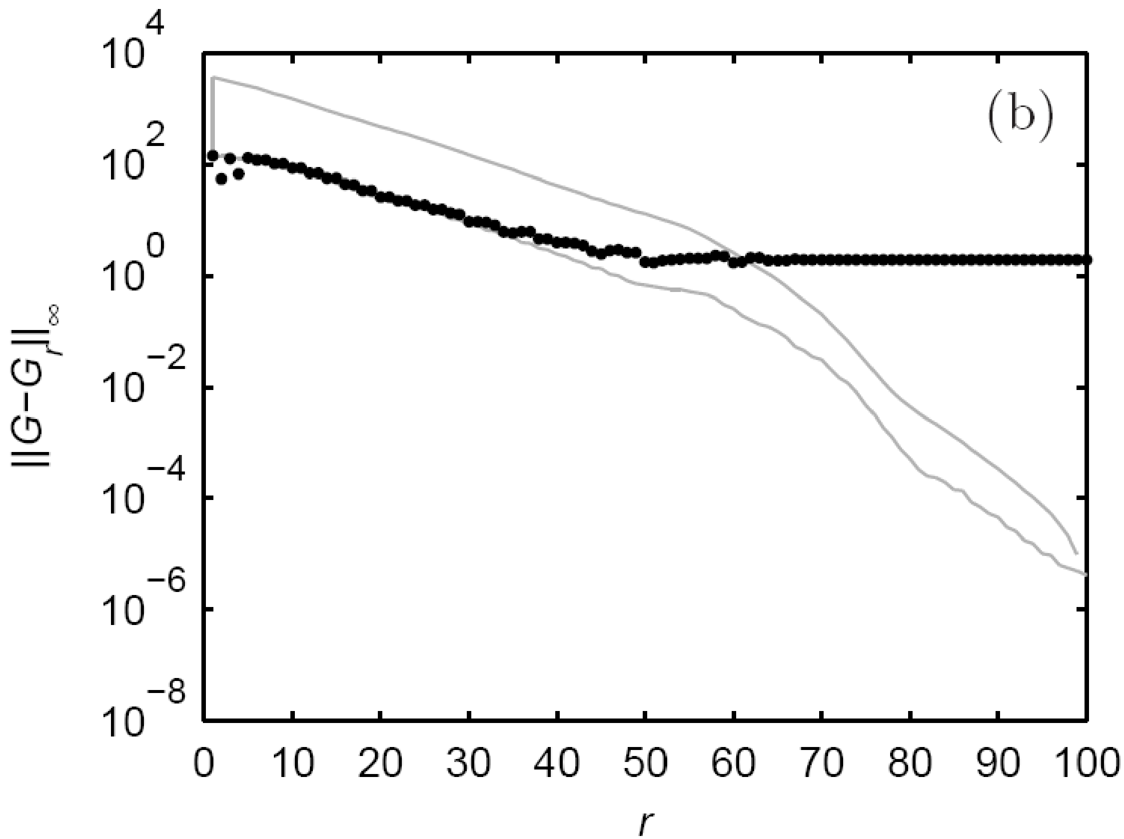
Model reduction error

- Frequency response
  - All inputs to all outpus
  - Order 2 capture the I/O behavior
  - Order 80 captures all frequencies



- Theoretical error bounds

$$\sigma_{r+1} \le \|G - G_r\|_\infty \le 2 \sum_{i=r+1}^{n} \sigma_i$$



Performance of controller

- Disturbance input white-noise
- Energy minimized at sensor 2

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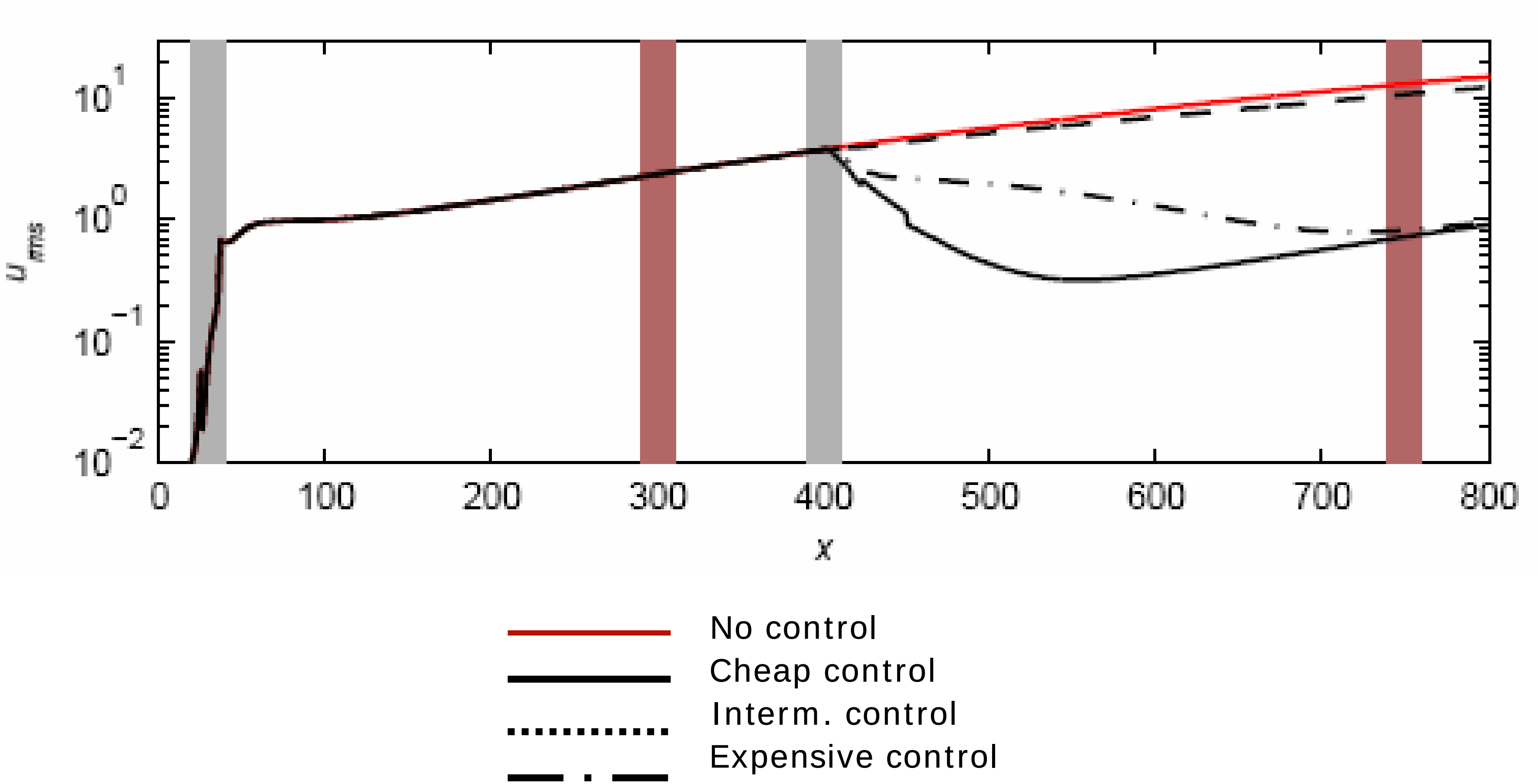


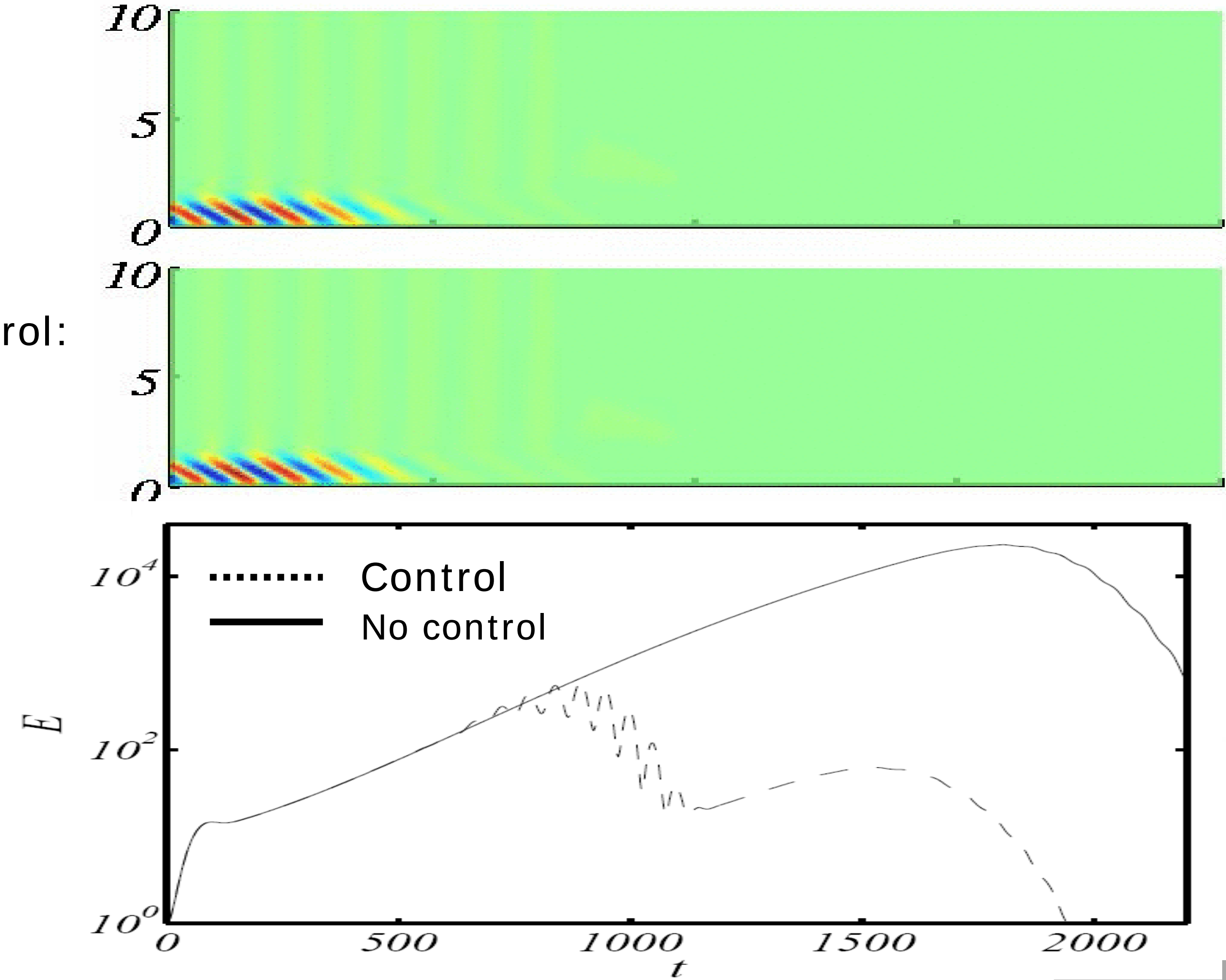
Control of disturbance

No control:

10^4 growth

Linear control:

10^2 growth



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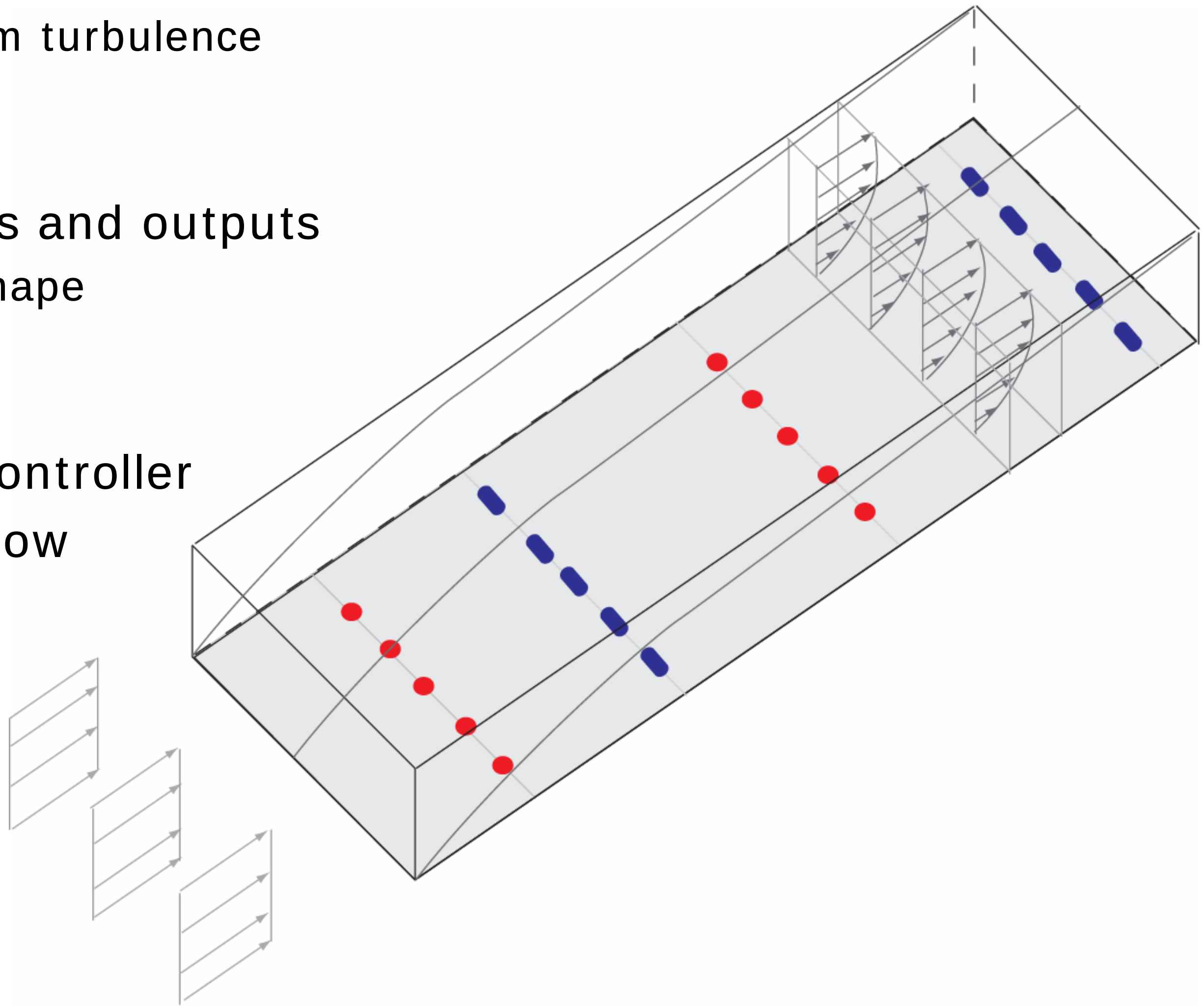
3D disturbances in flat-plate

- Disturbances:
  - Free-stream turbulence
  - Streaks
- Rows of inputs and outputs
  - Size and shape
  - Spacing
- Apply linear controller to nonlinear flow
- Robustness
- KTH/MTL windtunnel



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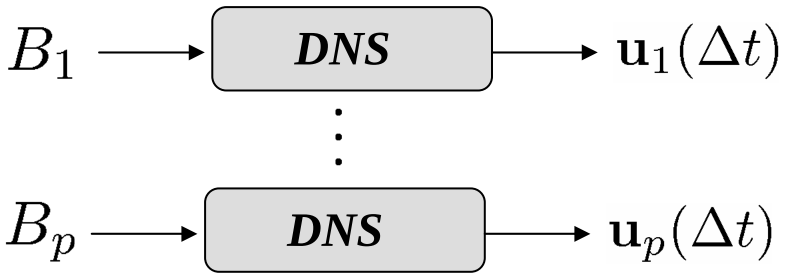
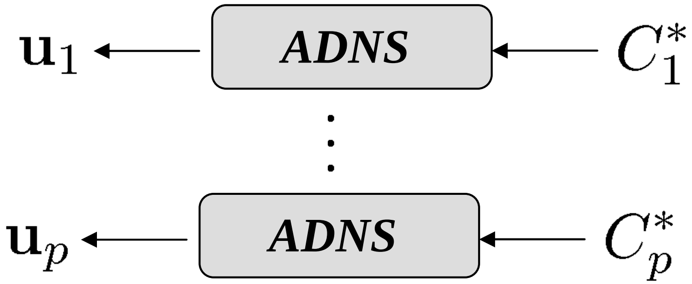
The snapshot method

- If number of inputs (p) and ouputs (r) are small, e.g. <10-50
- Extention: Output projection (Rowley, 2005) if either number of inputs or ouputs is large

$$\mathbf{X} = \{B_1, \dots, e^{A\Delta t}B_1, e^{A\Delta t}B_2, \dots, e^{A\Delta t}B_p\}$$



$$\mathbf{Y} = \{C_1^*, \dots, e^{A^*\Delta t}C_1^*, e^{A^*\Delta t}C_2^*, \dots, e^{A^*\Delta t}C_r^*\}$$



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3D balanced modes

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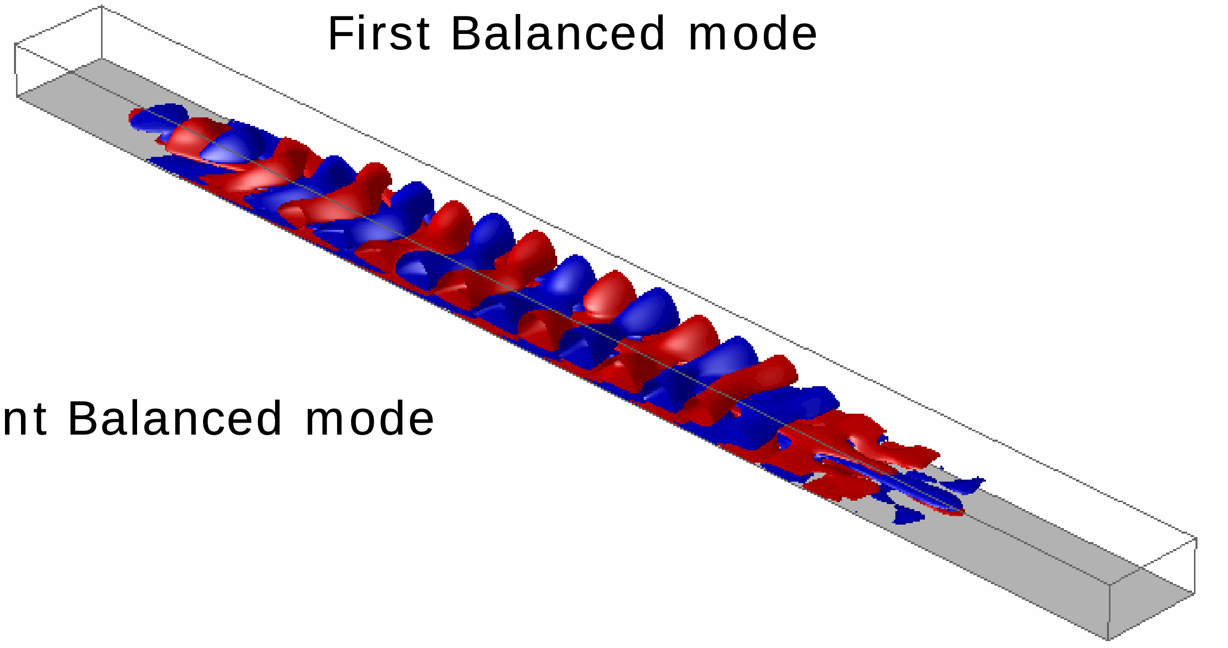
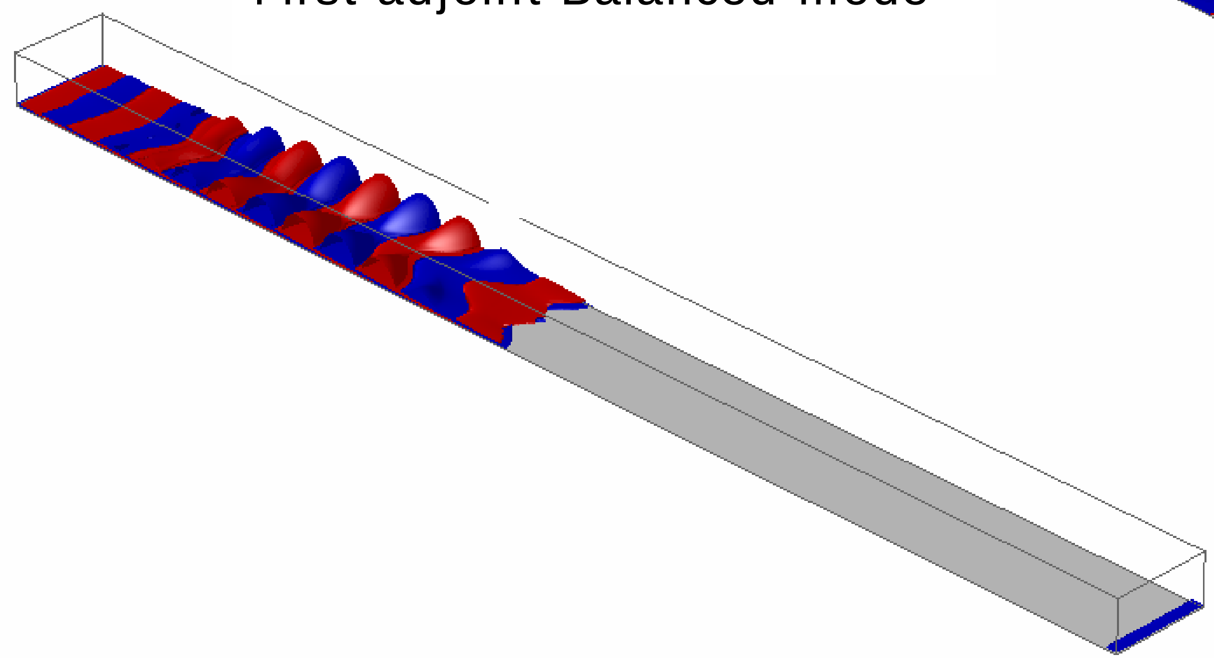




Balanced modes by iteratively techniques

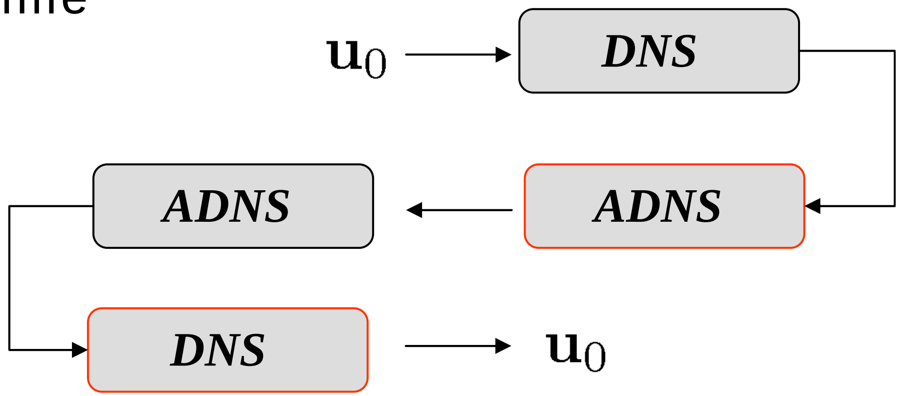
- The eigenvalue problem

$$PQ\mathbf{u}_j = \sigma_j \mathbf{u}_j$$

- Subspace

$$\mathcal{K} = \{\mathbf{u}_0, \int_0^{\Delta t} e^{A\tau} BB^* e^{A^*\tau} \mathrm{d}\tau \int_0^{\Delta t} e^{A^*\tau} C^* C e^{AT\tau} \mathrm{d}\tau \mathbf{u}_0, \ldots\}$$

- Basis vector snapshots of flow fields separated by constant time



- Backward to compute adjoint modes

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Conclusions

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- Existing CFD codes can be used for modern stability analysis and control design
- Computational cost of the same order as numerical simulations
- Swept wing, Seperated flows, Flows over steps and cavities, Flows in ducts and corners, Wake-vortex flows and bluff bodies

Jet in crossflow

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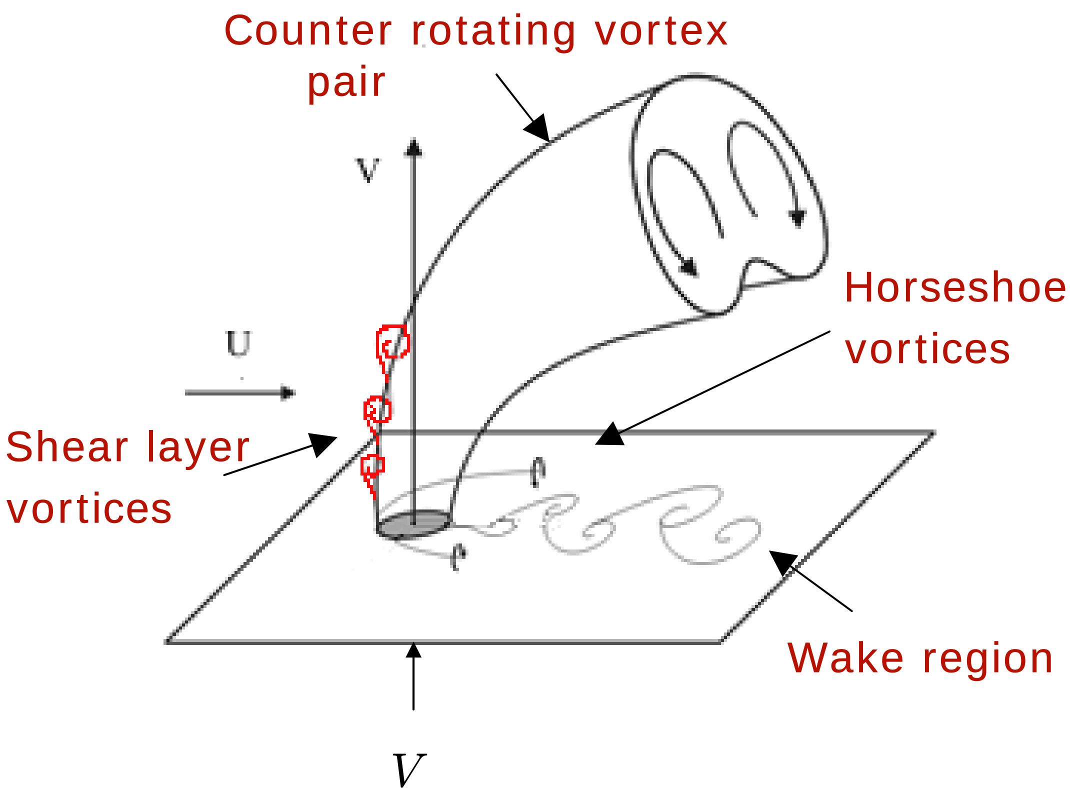

$$R = \frac{U}{V} = 3 \qquad \mathrm{Re} = \frac{U\delta_0}{\nu} = 165$$

Direct Numerical Simulation

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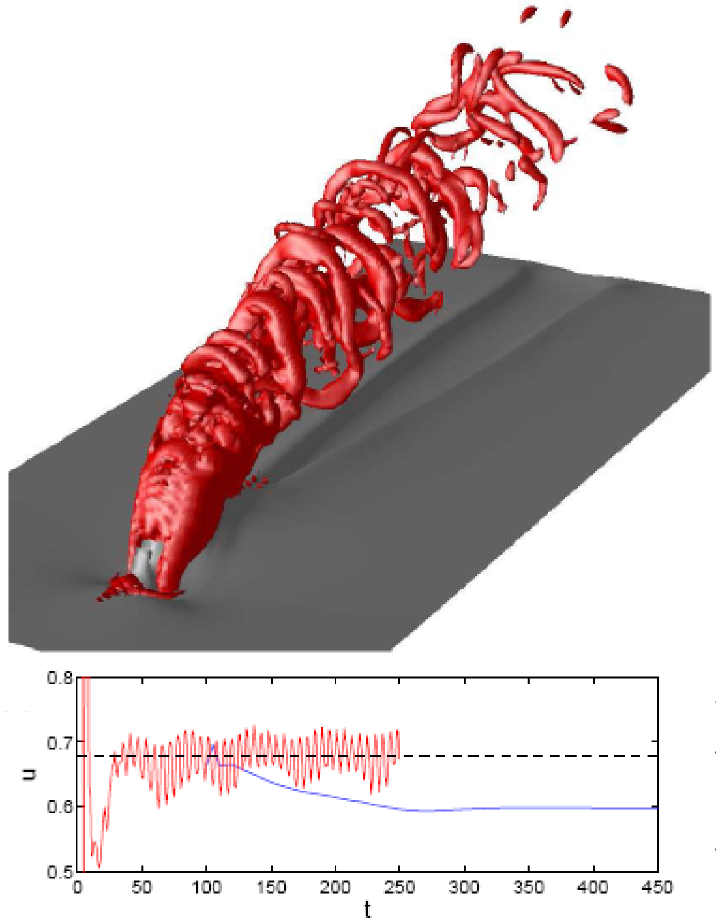
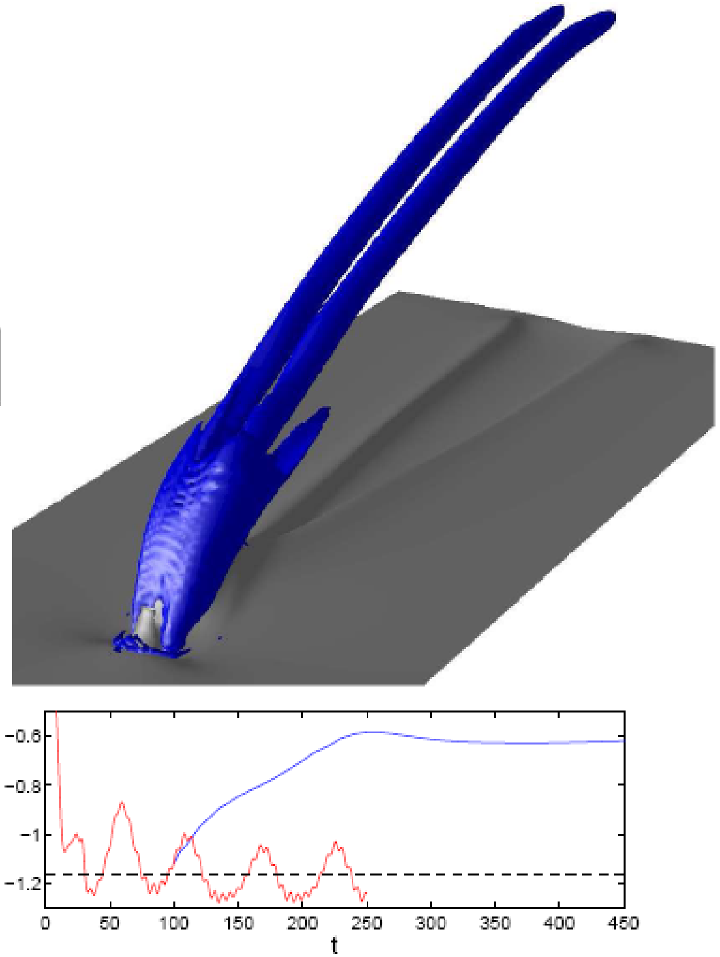
DNS simulation

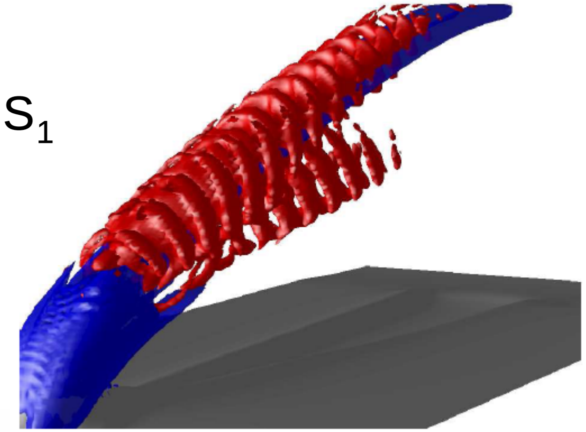


Steady-State (SFD)

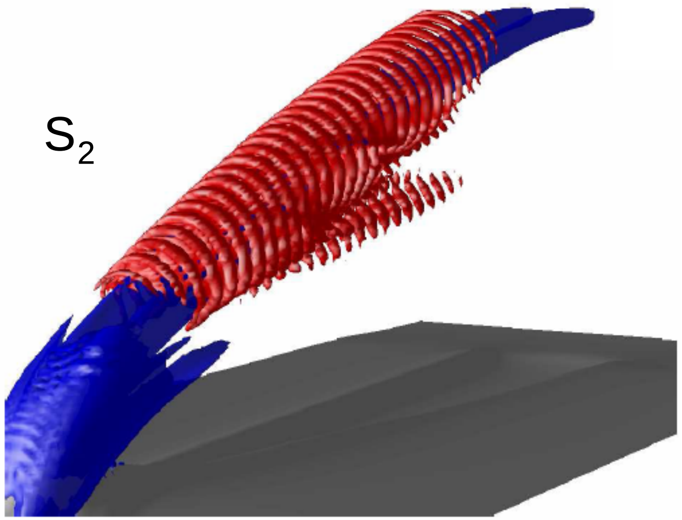


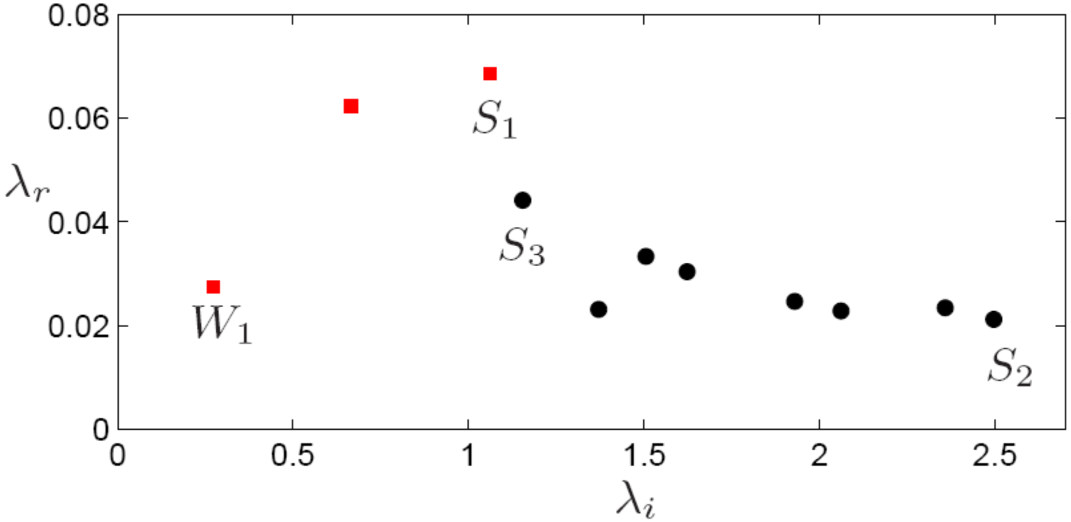
Fully 3D global eigenmodes

S$_1$

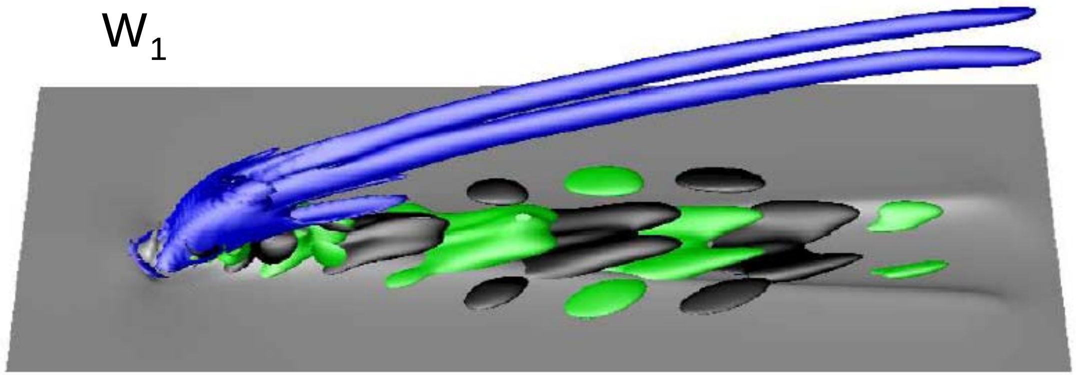


S$_2$





W$_1$



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Nekton 5000

- Spectral element code by Paul Fischer, Argonne National Laboratory
- 80,000 lines of f77 (some C)
- Structured grid – rectangular elements
- Curved geometries
- Massively parallel – 32 000 cores

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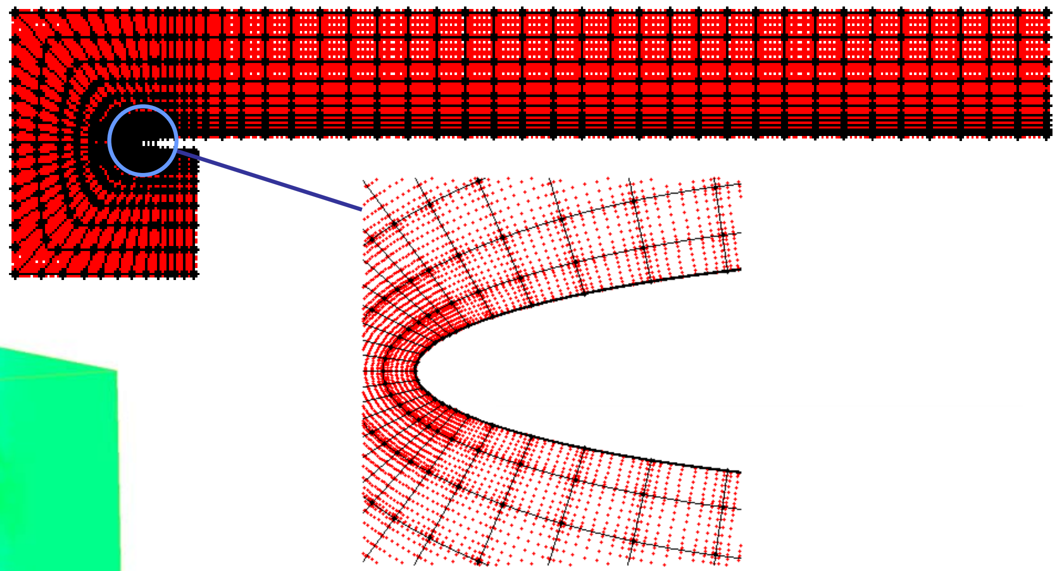

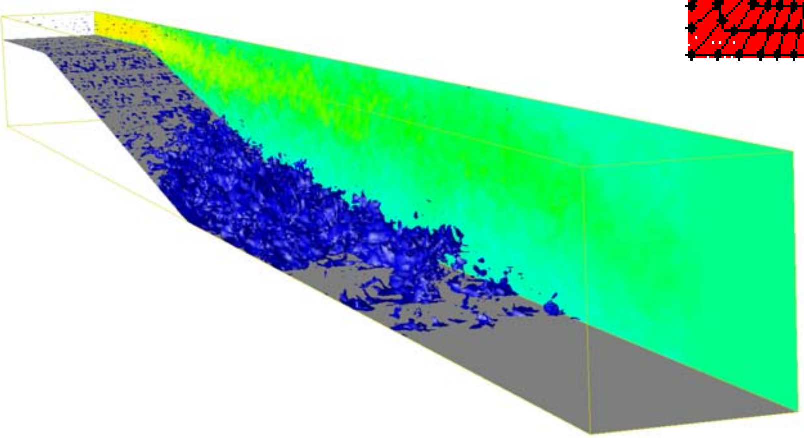

Extra slides

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