Stability analysis and control design of spatially developing flows

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Delaying transition

- Laminar flows are ordered and predictable
- Turbulent flows are chaotic and unpredictable
- **Drag-force** on surface is smaller for laminar than turbulent flows
Promoting transition

- Fluid injected through orifice (jet flow)

- Turbulent jets are more efficient in mixing jet fluid with ambient fluid than laminar flows
Stability analysis

- Behavior of small-amplitude disturbances in space and time

1. Receptivity
2. Disturbance behavior
3. Breakdown/transition
4. Turbulence
Control design

- Systematic approach
- Feedback control
- Model reduction
Outline

I. Stability analysis
   Global modes
   Optimal disturbances

II. Control design
   Model reduction
   Control performance

Spatially developing flows:
   Ginzburg-Landau equation (1D)
   Flat-plate boundary layer (2D)
   Jet in Crossflow (3D)
Stability analysis
Disturbance behavior

**Amplifiers:**
- Sensitive to disturbances
- Flat-plate, jets

**Oscillators:**
- Self-sustained oscillations
- Cylinder, hot/swirling jets, cavity

Large roughness height

Acoustic waves + small roughness height
Global approach

- Linearized Navier-Stokes equations

\[ \frac{\partial u}{\partial t} + (\mathbf{U} \cdot \nabla) u + (u \cdot \nabla) \mathbf{U} = -\nabla p + \frac{1}{\text{Re}} \nabla^2 u, \]
\[ \nabla \cdot u = 0, \]
\[ u(x, 0) = u_0, \]

- Initial value problem

\[ \frac{\partial u}{\partial t} = A u \]
\[ u(0) = u_0, \]

- Solution

\[ u(t) = e^{At} u_0 \]

- Investigate the properties of matrix exponential
- Matrix exponential is computationally expensive to evaluate
Dimension of discretized system

<table>
<thead>
<tr>
<th>Base Flow</th>
<th>Inhomogeneous direction(s)</th>
<th>Dimension of $u(t)$</th>
<th>Storage of $A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ginzburg-Landau</td>
<td>$U(x)$</td>
<td>1D</td>
<td>$10^2$</td>
</tr>
<tr>
<td>Blasius</td>
<td>$U(x,y)$</td>
<td>2D</td>
<td>$10^5$</td>
</tr>
<tr>
<td>Jet in crossflow</td>
<td>$U(x,y,z)$</td>
<td>3D</td>
<td>$10^7$</td>
</tr>
</tbody>
</table>

- Matrix $A$ very large for spatially developing flows
- Use Navier-Stokes solver (DNS) or any CFD code to approximate the action of exponential matrix:
  \[ u(t + T) = e^{A(t+T)} u(t) \]
- Time-stepper technique:
  - Never store matrices
  - Use only velocity fields at different times
Quest for eigenmodes

- Asymptotic behavior:
  \[ e^{At}u_j = \sigma_j u_j \]
  \[ |\sigma_1| > 1 \quad \text{globally unstable} \]
  \[ |\sigma_1| \leq 1 \quad \text{globally stable} \]
  \[ \rightarrow u_j \text{ called global eigenvectors} \]

- Transient behavior:
  \[ e^{A^*t}e^{At}u_j = \sigma_j u_j \]
  \[ |\sigma_1| > 1 \quad \text{convectively unstable} \]
  \[ |\sigma_1| \leq 1 \quad \text{convectively stable} \]
  \[ \rightarrow u_j \text{ called optimal disturbances} \]

- Time-stepper and iterative methods to compute modes
Previous work

- Global modes
  - Backward-facing step (Barkley et.al., 2002)
  - Flat-plate boundary layer (Ehrenstein et.al., 2005)
  - Smooth cavity (Åkervik et.al., 2007)
  - Cylinder wake (Gianetti & Luchini, 2007)
  - Recirculation bubble (Marquet et.al., 2008)

- Optimal disturbances
  - Swept Hiemens flow (Guegan et.al., 2007)
  - Backward-facing step (Barkley et.al., 2008)
  - Recirculation bubble (Marquet et.al., 2008)

- Current/future work: Jet in crossflow
Jet in crossflow

- Most of work experimental and no stability analysis
- Velocity Ratio
  \[ R = \frac{U}{V} = 3 \]
- Reynolds number
  \[ \text{Re} = \frac{U \delta_0}{\nu} = 165 \]
- Smoke stacks, film cooling etc.
Stability analysis – 4 steps

1. Simulate flow with DNS: Identify structures and regions
2. Compute baseflow: Steady-state solution
3. Compute impulse response of baseflow:
   \[ u(t) = e^{At}u_0 \]
4. Compute global modes of baseflow:
   \[ e^{At}u_j = \sigma_j u_j \]
Direct numerical simulations

- DNS: Fully spectral and parallelized
- Self-sustained global oscillations
- Probe 1 – shear layer
- Probe 2 – separation region
Impulse response

- Steady state computed using the SFD method (Åkervik et.al.)

- Asymptotic energy growth of perturbation
Global eigenmodes

- Global eigenmodes computed using ARPACK
- Growth rate: 0.08 \((\sigma_1 = 1.17)\)
- Strouhal number: 0.16

![Graph showing Perturbation energy and Global mode energy over time](image)
Control design
Flat-plate boundary layer

- 2D disturbances (TS-type/wavepackets)
- Convectively unstable (amplifier)
- Reynolds number:

\[ \text{Re} = \frac{U\delta_0}{\nu} = 1000 \]
Inputs and outputs

- **Inputs:**
  Roughness, free-stream turbulence, acoustic waves, blowing/suction etc.

- **Outputs:**
  Measurements of pressure, skin friction etc.

- **Setup for modern control design:**
  Minimize effects of disturbances on second sensor using first sensor and actuator
Control design – 5 steps

1. Construct plant: Flow, inputs, outputs
2. Construct reduced model from the plant using balanced truncation
3. Design controller using the reduced-model (LQG/H2)
4. Closed-loop: Connect sensor to actuator using the reduced controller
5. Run small controller online and evaluate closed-loop performance
Input-output behavior

- State-space formulation:
  \[ \dot{u} = Au + Bw \]
  \[ z = Cu. \]

- Solution:
  \[ z = \int_0^t C e^{A(t-\tau)} B w(\tau) \, d\tau \]

- \( B \) – Inputs
- \( C \) – Outputs
- \( w \) – Input signal
- \( z \) – Output signal
Model reduction

• Approximate the large system

\[ \dot{u} = Au + Bw \]
\[ z = Cu. \]

with a small system

\[ \dot{\hat{\kappa}} = \hat{A}\kappa + \hat{B}w \]
\[ \hat{z} = \hat{C}\kappa \]

so that the I/O behavior is preserved:

\[ \| z - \hat{z} \| \leq \epsilon \quad \forall w \]

• One systematic approach is balanced truncation (Moore 1981)
Balancing

- **Controllability**
  Flow states most easily excited by *input*
  Solution: **POD modes**
  Diagonalize the correlation matrix of the flow

\[ P = \int_0^\infty e^{A\tau} BB^* e^{A^*\tau} d\tau \]

- **Observability**
  Flow states that will most easily excite *output*
  Solution: **adjoint POD modes**
  Diagonalize correlation matrix adjoint system

\[ Q = \int_0^\infty e^{A^*\tau} C^* C e^{A\tau} d\tau \]

- **Balanced modes**
  Diagonalize both controllability and observability Gramian

\[ PQu_j = \sigma_j u_j \quad (n \times n), \quad n > 10^5 \]
Balanced modes

- Computed using snapshot method (Rowley 2005):
  Collect snapshots of forward and adjoint simulation + small eigenvalue problem

- Balanced modes (u-velocity):

- Oblique projection onto balanced modes to obtain reduced system (ROM):
  \[ A, B, C \rightarrow \hat{A}, \hat{B}, \hat{C} \]
Reduced system vs. Full system

Impulse response
- DNS: \( n = 10^5 \)
- ROM: \( m = 50 \)

Frequency response
- DNS: \( n = 10^5 \)
- ROM: \( m = 80 \)
- \( m = 50 \)
- \( m = 2 \)
Performance of controlled system

Control on

Control off

Noise

Sensor

Actuator

Objective

Disturbances  Sensor  Actuator  Sensor

(a)

(b)

(c)

(d)

0  1000  2000  3000  4000  5000  6000  7000  8000  9000  10000

0  5

0  20

0  5

0  100

0  -5

0  -20

0  -5

0  100

Performance of controlled system

- Disturbance amplitude efficiently damped
Summary

• **Time-stepper technique** necessary for modern stability analysis and control design of complex flow systems

• **Stability** – Both long and short time analysis possible using time-steppers

• **Model reduction** – Input-output is preserved using balanced truncation

• **Control design** – Using reduced-order models optimal/robust control schemes become applicable to complex flow systems
Outlook

- **Jet in crossflow**
  - Bifurcation analysis
  - Find critical velocity ratio
  - Sensitivity to forcing (adjoint global modes)
  - Optimal disturbances

- **Flat-plate boundary layer**
  - Three dimensional disturbance
  - Realistic actuators and sensors
  - Delay transition to turbulence