Linear Stability & Control of Fluid Flows: Coping with High-Dimensional Discretizations

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Aim of Presentation

- How extremely large eigenvalue problems (EVP) arise in stability analysis and control design of complex flows?
- How do we solve these large EVP using only snapshots of flow fields?
Outline

1. DNS and Linearized Equations: Problem Setup
   - Flow physics (Transition, Jet in Crossflow)
   - Governing equations
   - Global modes and transient growth

2. Applications to Flow Stability
   - Jet in cross flow
   - Blasius boundary layer

3. Flow Control
   - Model reduction by balanced truncation
   - Observability and controllability

4. Conclusions

Understanding & Controlling Transition to Turbulence

- Drag-force on surface is smaller for laminar than turbulent flows → Delay transition
- Turbulent jets are more efficient in mixing jet fluid with ambient fluid than laminar flows → Promote transition
Navier-Stokes Equations

- Nonlinear and 3D PDE

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u}
\]

\[\nabla \cdot \mathbf{u} = 0\]

\[\mathbf{u} = \mathbf{u}(x, y, z, t)\]

- Discretized, dynamical system:

\[\dot{\mathbf{u}} = f(\mathbf{u}), \quad \mathbf{u} \in \mathbb{R}^{10^5-10^7}\]

- Direct Numerical Simulation (DNS):

- Many efficient numerical codes exist for solving these equations

DNS: Transition on Flat-Plate

(decaying) freestream turbulence

high velocity

low velocity

contours of \(\lambda_2\)
How to Analyze the System?

- Enormous state-space:
  \[ \dot{u} = f(u), \quad u \in \mathbb{R}^{10^5-10^7} \]

- How do we?
  1. Find fixed points (steady-state, limit cycles, traveling waves etc.)
    \[ 0 = f(u_0) \]
  2. Analyze stability (global, local, short-time etc.)
    \[ \dot{u} = \nabla f(u_0) \cdot u \]
  3. Control/optimize the flow (disturbance, turbulence, shape etc.)
    \[ \dot{u} = f(u) + \phi \]

- With minimal modification of existing DNS codes?
**Timestepping Technique**

- **Forward problem (IVP)**
  \[
  \dot{u} = Au \\
  u(0) = u_0
  \]

- **Solution**
  \[
  u(t) = \exp(At)u_0 = T(t)u_0
  \]

- **Action is Navier-Stokes solver**

- **Adjoint problem**
  \[
  \dot{u} = A^*u \\
  u(T) = u_T
  \]

- **Solution (backward in time)**
  \[
  u(t) = \exp(A^*t)u_T = T^*(t)u_T
  \]

- **Action is adjoint Navier-Stokes solver**

**Asymptotic Response of IVP**

- **Asymptotic stability,**
  \[
  Av_j = \lambda_j v_j
  \]

- **Invariant under exponential transformation:**
  \[
  \exp(At)v_j = \exp(\lambda_j)t v_j
  \]

- **If exist unstable eigenvalue**
  \[
  E(t) = \|v\|^2 \to \infty \quad \text{as} \quad t \to \infty
  \]

- **Usually matrix A is non-normal, i.e.**
  \[
  \|VV^{-1}\| > 1
  \]
Short-Time Response of IVP

- Maximum amplification,
  \[ G(t) = \frac{E(t)}{E(0)} = \frac{\| \exp(At)u_0 \|^2}{\| u_0 \|^2} = \frac{\langle \exp(A^*t)\exp(At)u_0, u_0 \rangle}{\| u_0 \|^2} = \sigma_1 \]

- Short-time stability:
  \[ T^*(t)T(t)v_j = \sigma_j v_j \]

- If exist unstable eigenvalue, the disturbance grow at time \( t \):
  \[ G(t) = \sigma_1 > 1 \]

Iterative Techniques

- Global eigenmodes
  \[ T(t)v_j = \sigma_j v_j \]
- Krylov subspace
  \[ K = \{ u_0, T(\Delta t)u_0, \ldots \} \]
- Snapshots of flow fields separated by constant time
- Backwards to compute adjoint modes

- Optimal disturbances
  \[ T^*(t)T(t)v_j = \sigma_j v_j \]
- Krylov subspace
  \[ K = \{ u_0, T^*(\Delta t)T(\Delta t)u_0, \ldots \} \]
- Snapshots of adjoint fields separated constant time
- Modes orthogonal
Applications

- EVP1: Global eigenmodes
  - Application on fully 3D “Jet in Crossflow” (JCF)
  - Parallel ARPACK used with 2 millions d.o.f

- EVP2: Optimal disturbances
  - Application on 2D “flat-plate boundary layer” (Blasius)
  - ARPACK with $10^5$ d.o.f.

- EVP3: Balanced modes (not yet introduced)
  - “Flat-plate boundary layer”
  - Control design with inputs and outputs
  - The snapshot method

Direct Numerical Simulation

- Is the flow linearly globally unstable?

- DNS:
  \[ \dot{\mathbf{u}} = f(\mathbf{u}), \quad \mathbf{u} \in \mathcal{R}^{10^6} \]
Steady-State Solution

- Steady-solution to Navier-Stokes:
  \[ 0 = f(u_0) \]
  Selective frequency damping (SFD)
  Alternative to Newton iterations

- Steady structures:
  Horseshoe vortices
  Counter-rotating vortex pair


Streamwise velocity
\[ \lambda_2 \text{ Vortex identification criterion} \]

Spectrum of JCF

- Unstable eigenmodes of the linearized Navier-Stokes

\[ (EVP1) \quad T(t)v_j = \sigma_j v_j \quad \rightarrow \quad \lambda_j = \log(\sigma_j) \]
Global Eigenmodes of JCF

- Fully three dimensional
- Localized wavepackets wrapped around CVP

Streamwise velocity (baseflow)
\[ \lambda_2 \text{ Vortex (baseflow)} \]
\[ \lambda_3 \text{ Vortex (global mode)} \]

Global Eigenmodes of JCF

- Low frequency mode
- Wake instability

Streamwise velocity (baseflow)
\[ \lambda_2 \text{ Vortex (baseflow)} \]
Spanwise velocity (global mode)
Stability Analysis

- Behavior of small-amplitude disturbances in space and time
- 2D baseflow $\rightarrow 10^5$ d.o.f.

Global Eigenmodes of Blasius

- All eigenmodes of the linearized Navier-Stokes stable

$$ (EVP1) \quad T(t) \mathbf{v}_j = \sigma_j \mathbf{v}_j \quad \rightarrow \quad \lambda_j = \log(\sigma_j) $$
Optimal Disturbances

- Many unstable modes

\[ T^*(t)T(t)v_j = \sigma_j v_j \] (EVP2)

Summary

- Jet-in-Crossflow:
  - Asymptotically unstable
  - Self-sustained oscillations

- Flat-plate:
  - Only short-time growth
  - Noise amplifier
Control Design

- Take measurements and adjust actuation accordingly
- Account for unknown variations:
  - Sensor noise
  - Modeling errors

Model Reduction

- Approximate the large system
  \[
  \dot{u} = Au + Bw, \\
  z = Cu + Dw
  \]
  \(n > 10^5\)
  
  with a small system
  \[
  \dot{\hat{u}} = A_r \hat{u} + B_r w, \\
  \hat{z} = C_r \hat{u} + D_r w
  \]
  \(r < 100\)

  so that the I/O behavior is preserved:
  \[
  \sup_w \frac{\|z - \hat{z}\|_2}{\|w\|_2} = \|G - G_r\|_\infty \leq \epsilon(r)
  \]

- One systematic approach is balanced truncation (Moore 1981)

  \[
  \sigma_{r+1} \leq \|G - G_r\|_\infty \leq 2 \sum_{i=r+1}^{n} \sigma_i
  \]
Controllability

- Flow states most easily excited by input
- Diagonalize the correlation matrix of the flow
  \[ P = \int_0^\infty e^{A^* B B^* e^{A^*} d \tau} \]
- POD modes

Observability

- Flow states that will most easily excite output
- Diagonalize the correlation matrix of adjoint flow
  \[ Q = \int_0^\infty e^{A^*} C^* C e^{A^*} d \tau \]
- Adjoint POD modes
Balancing

- Controllability
  \[ P = \int_0^\infty e^{A^T \tau} B B^* e^{A \tau} d\tau \]

- Observability
  \[ Q = \int_0^\infty e^{A^T \tau} C^* C e^{A \tau} d\tau \]

- Balanced modes
  Diagonalize both controllability and observability Gramian
  \[ PQ u_j = \sigma_j^2 u_j \quad (n \times n), \quad n > 10^5 \]

Controllable Subspace

- Construct dominant controllable subspace
  \[ X = \{ B_1, \ldots, e^{A \Delta t} B_1, e^{A \Delta t} B_2, \ldots, e^{A \Delta t} B_p \} \]

- Basis vector snapshots of flow fields separated by constant time
  \[ B_1 \xrightarrow{\text{DNS}} u_1(\Delta t), \quad \ldots, \quad B_p \xrightarrow{\text{DNS}} u_p(\Delta t) \]

- Approximate Gramian
  \[ P \approx X X^T \]
Observable Subspace

- Construct dominant observable subspace
  \[ Y = \{ C_1^*, \ldots, e^{A^*\Delta t}C_1^*, e^{A^*\Delta t}C_2^*, \ldots, e^{A^*\Delta t}C_p^* \} \]

- Basis vector snapshots of adjoint flow fields separated by constant time
  \[
  \begin{align*}
  &u_1 \quad \text{ADNS} \quad C_1^* \\
  &\vdots \\
  &u_p \quad \text{ADNS} \quad C_p^*
  \end{align*}
  \]

- Approximate Gramian
  \[ Q \approx YY^T \]

The Snapshot Method

- Original very large EVP:
  \[ PQ \approx XX^TVV^T \]

- Expand modes in snapshots:
  \[ T = (u_1, \ldots, u_m) = XV \]

- Singular value decomposition of matrix (#snapshot x # snapshots):
  \[ Y^T X = U\Sigma V^T \]

Sirovich (1987)
Rowley (2005)
Balanced Modes

- Balanced modes
- Adjoint balanced modes

\[ T = XV \]

\[ S = YU \]

- Bi-orthogonal: \( S^T T = I \)
- Galerkin projection to obtained reduced-order model

Performance of Reduced System

- Disturbance → Sensor
- Actuator → Objective
- Disturbance → Objective

- DNS: \( n = 10^5 \)
- ROM: \( m = 50 \)
Control of Disturbance

Conclusions

- Existing CFD codes can be used for modern stability analysis and control design
- Computational cost of the same order as numerical simulations
- Swept wing, Separated flows, Flows over steps and cavities, Flows in ducts and corners, Wake-vortex flows and bluff bodies