Linear Stability & Control of Fluid Flows:
Coping with High-Dimensional Discretizations

Linné Flow Centre
KTH Mechanics

Shervin Bagheri
Linné Flow Centre, KTH Mechanics
Stockholm, Sweden

Collaborators:
Espen Åkervik, Philipp Schlatter,
Luca Brandt and Dan Henningson

Rice University, Houston, Texas,
November 26, 2008

Aim of Presentation

Linné Flow Centre
KTH Mechanics

- How extremly large eigenvalue problems (EVP) arise in stability analysis and control design of complex flows?
- How do we solve these large EVP using only snapshots of flow fields?

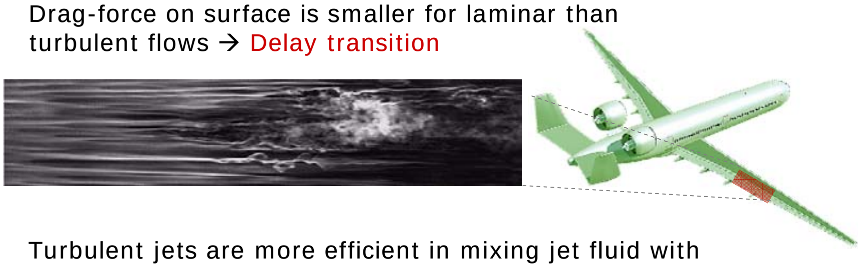
Outline

- 1. DNS and Linearized Equations: Problem Setup
  – Flow physics (Transition, Jet in Crossflow)
  – Governing equations
  – Global modes and transient growth
- 2. Applications to Flow Stability
  – Jet in cross flow
  – Blasius boundary layer
- 3. Flow Control
  – Model reduction by balanced truncation
  – Observability and controllability
- 4. Conclusions

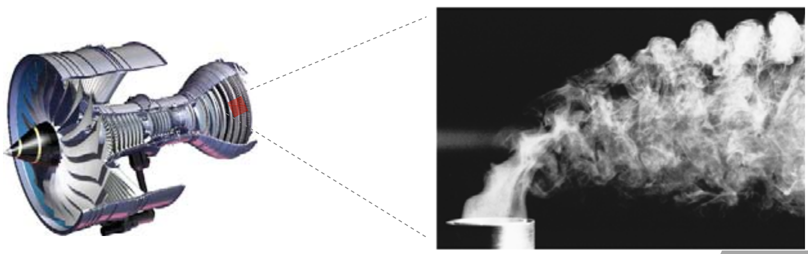
Linné Flow Centre
KTH Mechanics

Understanding & Controlling Transition to Turbulence

- Drag-force on surface is smaller for laminar than turbulent flows → Delay transition



- Turbulent jets are more efficient in mixing jet fluid with ambient fluid than laminar flows → Promote transition



Linné Flow Centre
KTH Mechanics

Navier-Stokes Equations

Linné Flow Centre
KTH Mechanics

- Nonlinear and 3D PDE

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2\mathbf{u}$$
$$\nabla\cdot\mathbf{u} = 0$$

$$\mathbf{u} = \mathbf{u}(x,y,z,t)$$

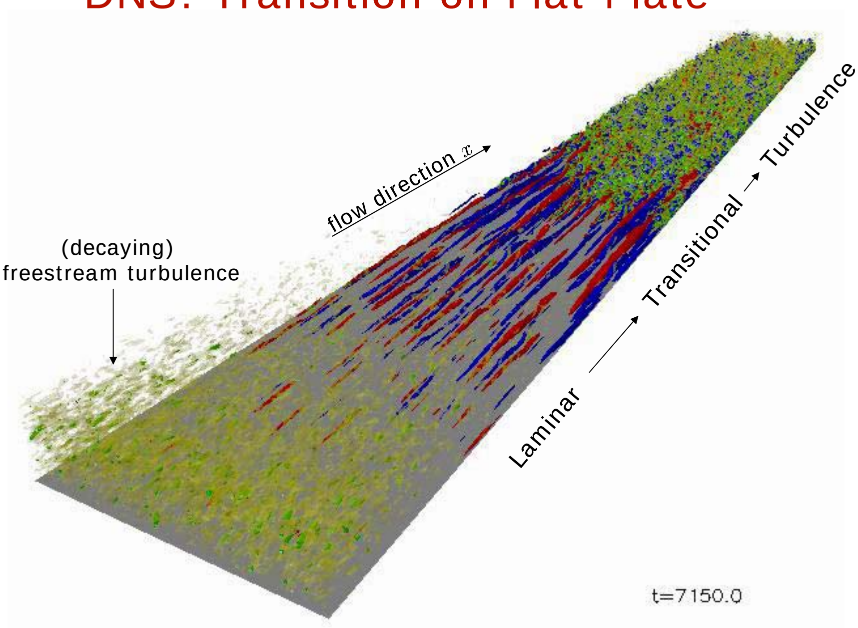
- Discretized, dynamical system:

$$\dot{\mathbf{u}} = f(\mathbf{u}), \qquad \mathbf{u} \in \mathcal{R}^{10^5-10^7}$$

- Direct Numerical Simulation (DNS):
- Many efficient numerical codes exits for solving these equations

DNS: Transition on Flat-Plate

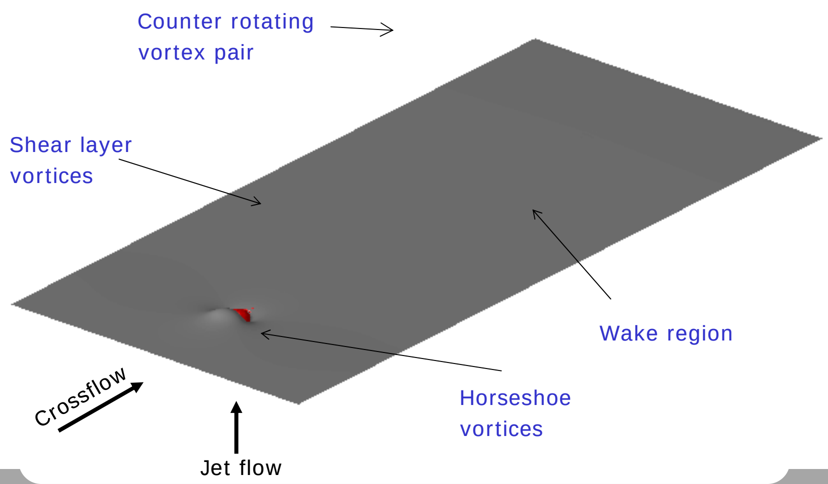
Linné Flow Centre
KTH Mechanics



high velocity
low velocity
contours of λ_2

DNS: Jet In Crossflow

Linné Flow Cen
KTH Mechani



u-velocity
contours of λ_2

How to Analyze the System?

Linné Flow Centre
KTH Mechanics

- Enormous state-space:

$$\dot{\mathbf{u}} = f(\mathbf{u}), \qquad \mathbf{u} \in \mathcal{R}^{10^5 - 10^7}$$

- How do we?
  1. Find fixed points (steady-state, limit cycles, traveling waves etc..)

  $$0 = f(\mathbf{u}_s)$$

  2. Analyze stability (global, local, short-time etc..)

  $$\dot{\mathbf{u}} = \underbrace{\nabla(f(\mathbf{u}_s))}_{A} \mathbf{u}$$

  3. Control/optimize the flow (disturbance, turbuluence, shape etc..)

  $$\dot{\mathbf{u}} = f(\mathbf{u}) + \phi$$

- With minimal modification of existing DNS codes?

Timestepping Technique

Linné Flow Centre
KTH Mechanics

- Forward problem (IVP)

$$\dot{\mathbf{u}} = A\mathbf{u}$$
$$\mathbf{u}(0) = \mathbf{u}_0$$

- Solution

$$\mathbf{u}(t) = \exp(At)\mathbf{u}_0 = T(t)\mathbf{u}_0$$

- Action is Navier-Stokes solver

- Adjoint problem

$$\dot{\mathbf{u}} = A^*\mathbf{u}$$
$$\mathbf{u}(T) = \mathbf{u}_T$$

- Solution (backward in time)

$$\mathbf{u}(t) = \exp(A^*t)\mathbf{u}_T = T^*(t)\mathbf{u}_T$$

- Action is adjoint Navier-Stokes solver

Barkely et al (2006,2008)

Asymptotic Response of IVP

Linné Flow Centre
KTH Mechanics

- Asymptotic stability,

$$A\mathbf{v}_j = \lambda_j \mathbf{v}_j$$

- Invariant under exponential transformation:

$$\boxed{\exp(At)\mathbf{v}_j = \exp(\lambda_j)\mathbf{v}_j}$$

- If exist unstable eigenvalue

$$E(t) = \|\mathbf{v}\|^2 \to \infty \qquad \text{as} \qquad t \to \infty$$

- Usually matrix A is non-normal, i.e.

$$\|\mathbf{V}\mathbf{V}^{-1}\| > 1$$

Short-Time Response of IVP

Linné Flow Centre
KTH Mechanics

- Maximum amplification,

$$G(t) = \frac{E(t)}{E(0)} = \frac{\|\exp(At)\mathbf{u}_0\|^2}{\|\mathbf{u}_0\|^2} = \frac{\langle \exp(A^*t)\exp(At)\mathbf{u}_0, \mathbf{u}_0\rangle}{\|\mathbf{u}_0\|^2} = \sigma_1$$

- Short-time stability:

$$T^*(t)T(t)\mathbf{v}_j = \sigma_j \mathbf{v}_j$$

- If exist unstable eigenvalue, the disturbance grow at time t:

$$G(t) = \sigma_1 > 1$$

Iterative Techniques

Linné Flow Centre
KTH Mechanics

- Global eigenmodes

$$T(t)\mathbf{v}_j = \sigma_j \mathbf{v}_j$$

- Krylov subspace

$$\mathcal{K} = \{\mathbf{u}_0, T(\Delta t)\mathbf{u}_0, \ldots\}$$

- Snapshots of flow fields separated by constant time

- Backwards to compute adjoint modes

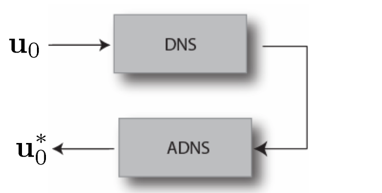
- Optimal disturbances

$$T^*(t)T(t)\mathbf{v}_j = \sigma_j \mathbf{v}_j$$

- Krylov subspace

$$\mathcal{K} = \{\mathbf{u}_0, T^*(\Delta t)T(\Delta t)\mathbf{u}_0, \ldots\}$$

- Snapshots of adjoint fields separated constant time



- Modes orthogonal

Applications

Linné Flow Centre
KTH Mechanics

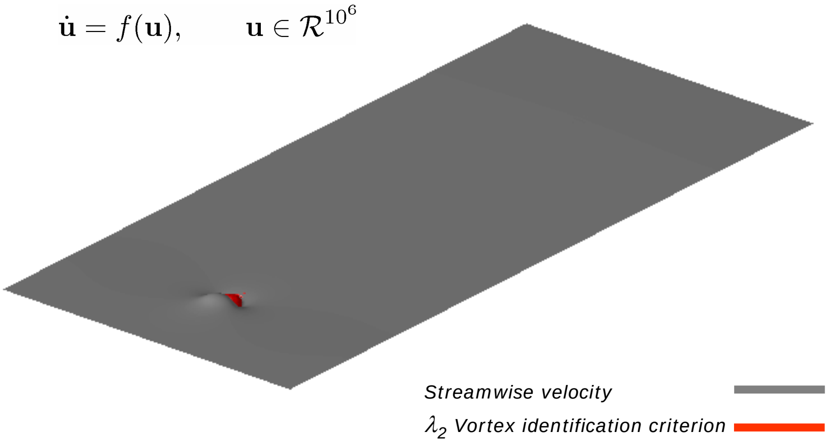
- EVP1: Global eigenmodes
  - Application on fully 3D ”Jet in Crossflow” (JCF)
  - Parallel ARPACK used with 2 millions d.o.f
- EVP2: Optimal disturbances
  - Application on 2D ”flat-plate boundary layer” (Blasius)
  - ARPACK with 10^5 d.o.f.
- EVP3: Balanced modes (not yet introduced)
  - ”Flat-plate boundary layer”
  - Control design with inputs and outputs
  - The snapshot method

Direct Numerical Simulation

Linné Flow Cen
KTH Mechanic

- Is the flow linearly globally unstable?
- DNS:

$$\dot{\mathbf{u}} = f(\mathbf{u}), \qquad \mathbf{u} \in \mathcal{R}^{10^6}$$



Steady-State Solution

- Steady-solution to Navier-Stokes:

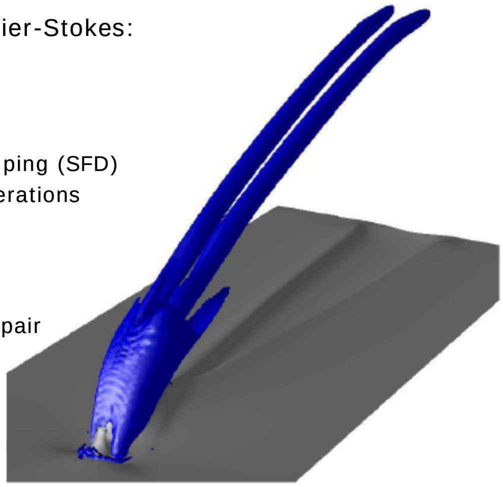
$$0 = f(\mathbf{u}_s)$$

Selective frequency damping (SFD)
Alternative to Newton iterations

- Steady structures:
  Horseshoe vortices
  Counter-rotating vortex pair

Linné Flow Centre
KTH Mechanics

SFD:
Åkervik et al (2006)



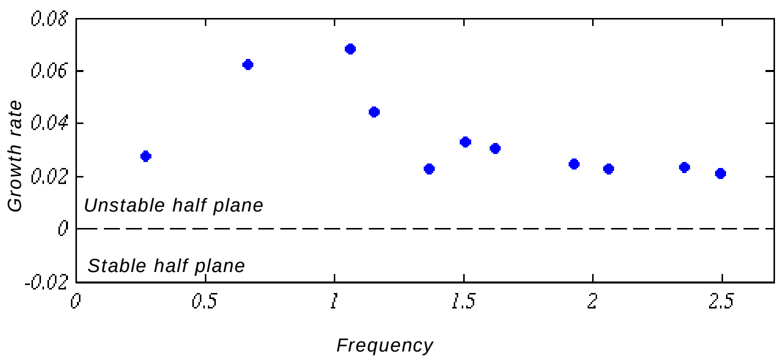
Streamwise velocity
λ_2 *Vortex identification criterion*

Spectrum of JCF

- Unstable eigenmodes of the linearized Navier-Stokes

(EVP1) $$T(t)\mathbf{v}_j = \sigma_j \mathbf{v}_j \longrightarrow \lambda_j = \log(\sigma_j)$$

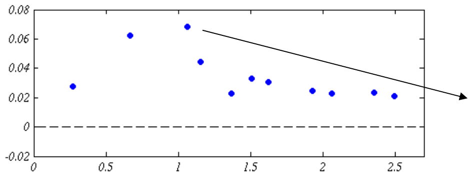
Linné Flow Centr
KTH Mechanics

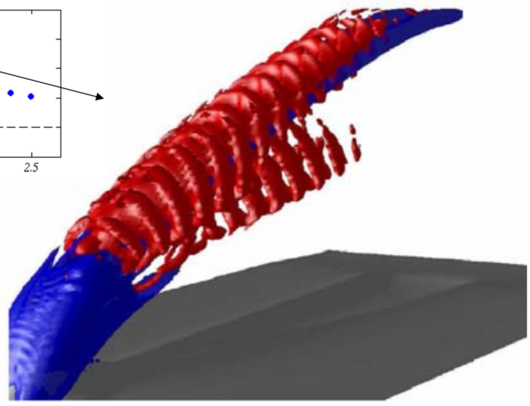


Global Eigenmodes of JCF

- Fully three dimensional
- Localized wavepackets wrapped around CVP

Linné Flow Centr
KTH Mechanics



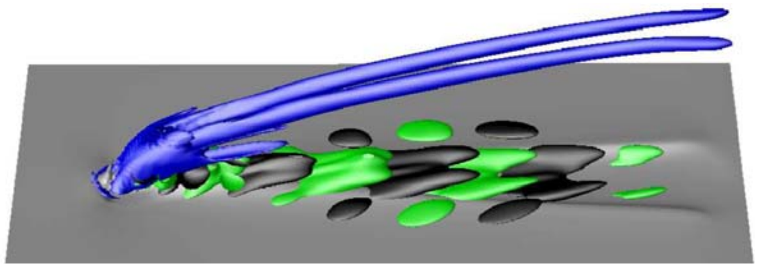


Streamwise velocity (baseflow)
λ_2 Vortex (baseflow)
λ_2 Vortex (global mode)

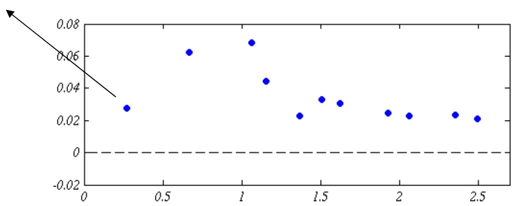
Global Eigenmodes of JCF

- Low frequency mode
- Wake instability

Linné Flow Centre
KTH Mechanics

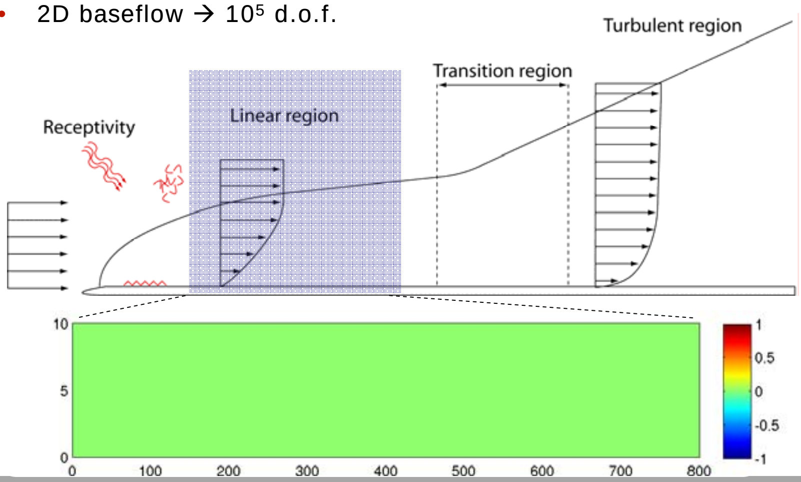


Streamwise velocity (baseflow)
λ_2 Vortex (baseflow)
Spanwise velocity (global mode)



Stability Analysis

- Behavior of small-amplitude disturbances in space and time
- 2D baseflow → 10^5 d.o.f.

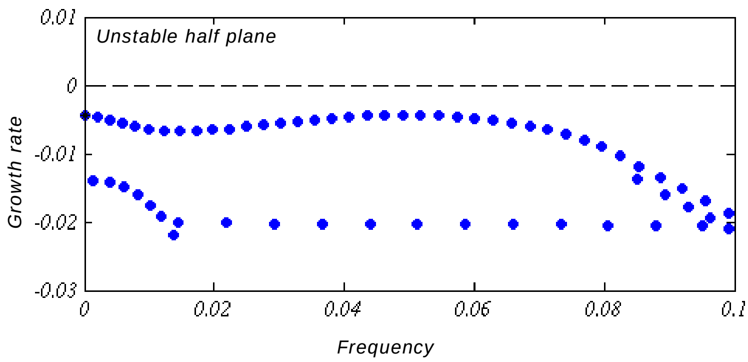


Linné Flow Centre
KTH Mechanics

Global Eigenmodes of Blasius

- All eigenmodes of the linearized Navier-Stokes stable

(EVP1) $$T(t)\mathbf{v}_j = \sigma_j \mathbf{v}_j \longrightarrow \lambda_j = \log(\sigma_j)$$

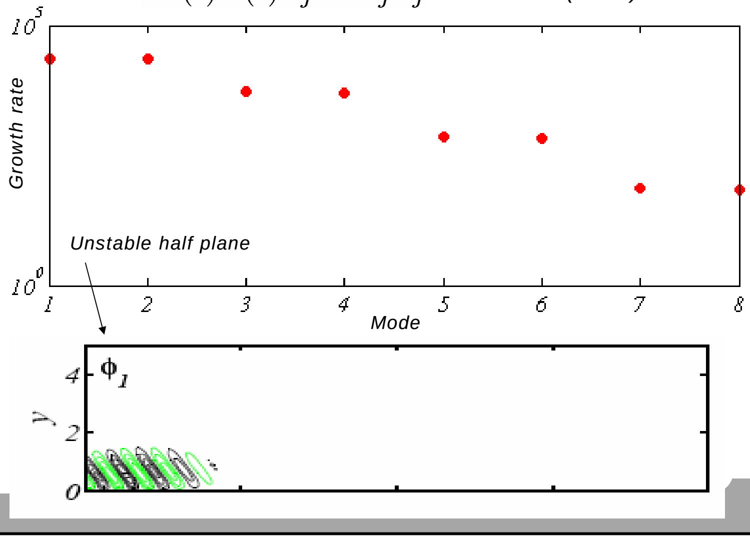


Linné Flow Centre
KTH Mechanics

Optimal Disturbances

- Many unstable modes

$$T^*(t)T(t)\mathbf{v}_j = \sigma_j \mathbf{v}_j \qquad \textit{(EVP2)}$$

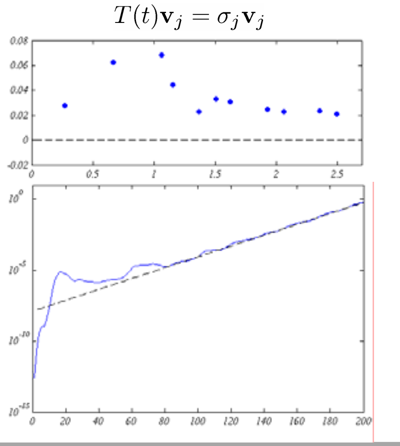


Linné Flow Centre
KTH Mechanics

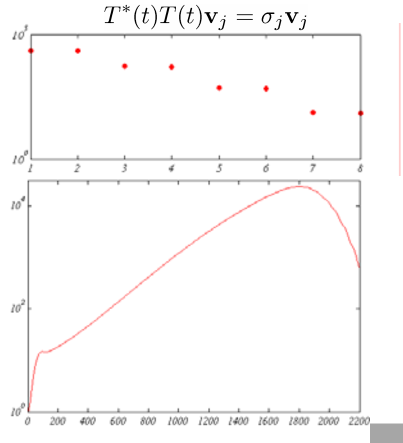
Summary

- Jet-in-Crossflow:
  - Asymptotically unstable
  - Self-sustained oscillations
- Flat-plate:
  - Only short-time growth
  - Noise amplifier

$$T(t)\mathbf{v}_j = \sigma_j \mathbf{v}_j$$

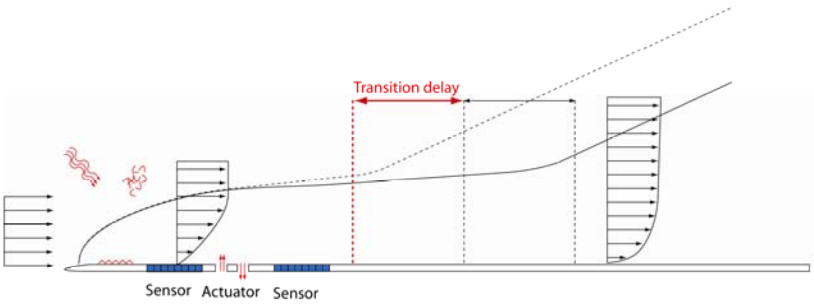


$$T^*(t)T(t)\mathbf{v}_j = \sigma_j \mathbf{v}_j$$



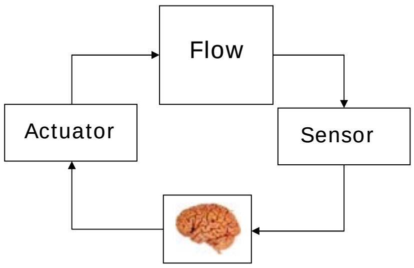
Linné Flow Centre
KTH Mechanics

Control Design



Linné Flow Centre
KTH Mechanics

- Take measurements and adjust actuation accordingly
- Account for unknown variations:
  Sensor noise
  Modeling errors



Model Reduction

Linné Flow Centre
KTH Mechanics

- Approximate the large system

$$\begin{aligned} \dot{\mathrm{u}} &= A\mathrm{u} + B\mathrm{w}, \\ \mathrm{z} &= C\mathrm{u} + D\mathrm{w} \end{aligned} \qquad n > 10^5$$

with a small system

$$\begin{aligned} \dot{\hat{u}} &= A_r\hat{u} + B_r\mathrm{w}, \\ \hat{\mathrm{z}} &= C_r\hat{u} + D\mathrm{w} \end{aligned} \qquad r < 100$$

so that the I/O behavior is preserved:

$$\sup_{\mathrm{w}} \frac{\|\mathrm{z} - \hat{\mathrm{z}}\|_2}{\|\mathrm{w}\|_2} = \|G - G_r\|_\infty \leq \epsilon(r)$$

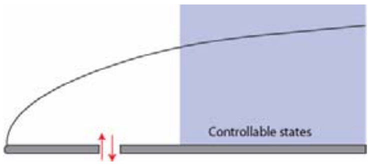
- One systematic approach is *balanced truncation* *(Moore 1981)*

$$\sigma_{r+1} \leq \|G - G_r\|_\infty \leq 2 \sum_{i=r+1}^{n} \sigma_i$$

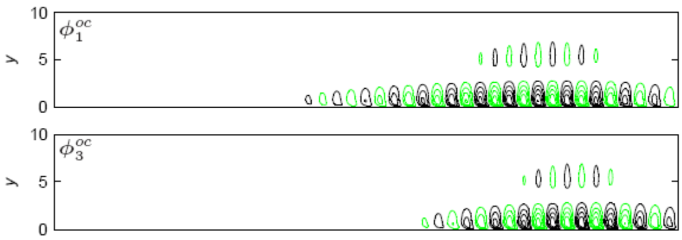
Controllability

- Flow states most easily excited by input
- Diagonalize the correlation matrix of the flow

$$P = \int_0^\infty e^{A\tau} B B^* e^{A^*\tau} \mathrm{d}\tau$$



- POD modes

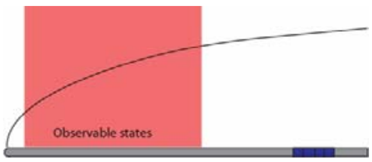


Linné Flow Centre
KTH Mechanics

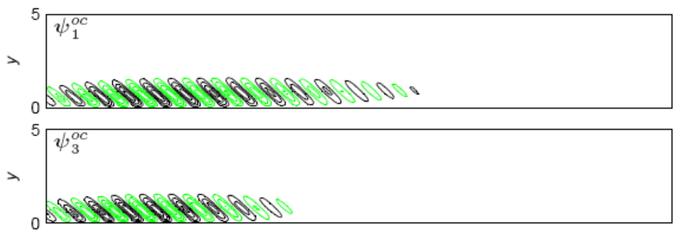
Observability

- Flow states that will most easily excite output
- Diagonalize the correlation matrix of adjoint flow

$$Q = \int_0^\infty e^{A^*\tau} C^* C e^{A\tau} \mathrm{d}\tau$$



- Adjoint POD modes



Linné Flow Centre
KTH Mechanics

Balancing

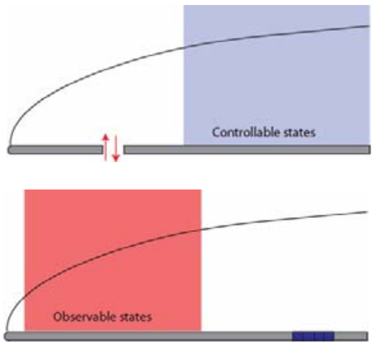
Linné Flow Centre
KTH Mechanics

- Controllability

$$P = \int_0^\infty e^{A\tau} B B^* e^{A^*\tau} \mathrm{d}\tau$$

- Observability

$$Q = \int_0^\infty e^{A^*\tau} C^* C e^{A\tau} \mathrm{d}\tau$$



- Balanced modes
  Diagonalize both controllability and observability Gramian

$$PQ\mathbf{u}_j = \sigma_j^2 \mathbf{u}_j \qquad (n \times n), \qquad n > 10^5$$

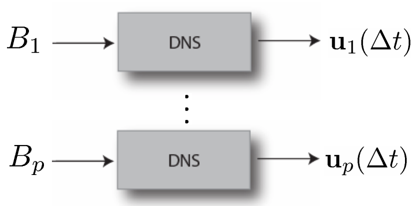
Controllable Subspace

Linné Flow Centre
KTH Mechanics

- Construct dominant controllable subspace

$$\mathbf{X} = \{B_1, \ldots, e^{A\Delta t} B_1, e^{A\Delta t} B_2, \ldots, e^{A\Delta t} B_p\}$$

- Basis vector snapshots of flow fields separated by constant time



- Approximte Gramian

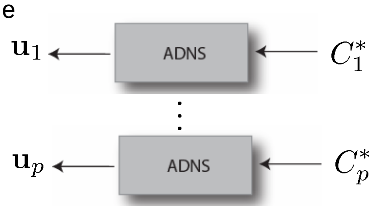
$$P \approx X X^T$$

Observable Subspace

- Construct dominant observable subspace

$$\mathbf{Y} = \{C_1^*, \dots, e^{A^*\Delta t}C_1^*, e^{A^*\Delta t}C_2^*, \dots, e^{A^*\Delta t}C_r^*\}$$

- Basis vector snapshots of adjoint flow fields separated by constant time



- Approximte Gramian

$$Q \approx YY^T$$

Linné Flow Centre
KTH Mechanics

The Snapshot Metod

- Original very large EVP:

$$\underbrace{PQ}_{\approx XX^TYY^T} \quad \mathbf{u}_j = \sigma_j \mathbf{u}_j \qquad (n \times n), \qquad n > 10^5$$

- Expand modes in snapshots:

$$\mathbf{T} = (\mathbf{u}_1, \dots, \mathbf{u}_m) = \mathbf{X}V$$

- Singular value decomposition of matrix (#snapshot x # snapshots):

$$\mathbf{Y}^T\mathbf{X} = U\Sigma V^T \qquad (m \times m), \qquad m \approx 10^2$$

Linné Flow Centre
KTH Mechanics

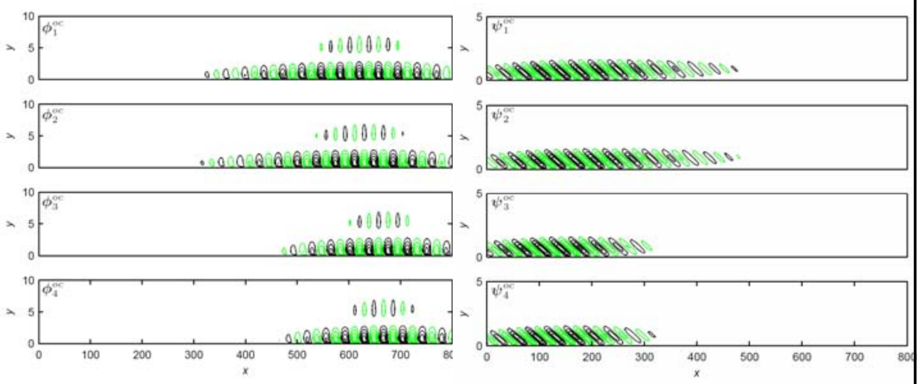
Sirovich (1987)
Rowley (2005)

Balanced Modes

Linné Flow Centre
KTH Mechanics

- Balanced modes — Adjoint balanced modes

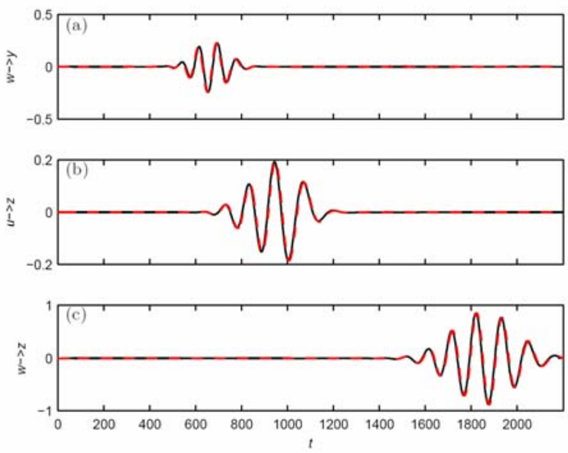
$\mathbf{T} = \mathbf{X}\mathbf{V}$ — $\mathbf{S} = \mathbf{Y}\mathbf{U}$



- Bi-orthogonal: $\mathbf{S}^{*}\mathbf{T} = I$
- Galerkin projection to obtained reduced-order model

Performance of Reduced System

Linné Flow Centre
KTH Mechanics



Disturbance ⟶ Sensor

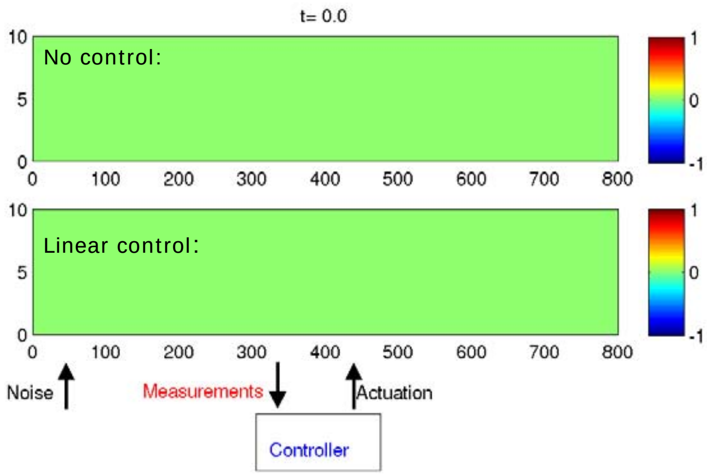
Actuator ⟶ Objective

Disturbance ⟶ Objective

DNS: $n=10^5$

ROM: $m=50$

Control of Disturbance



Linné Flow Centre
KTH Mechanics

Conclusions

- Existing CFD codes can be used for modern stability analysis and control design
- Computational cost of the same order as numerical simulations
- Swept wing, Seperated flows, Flows over steps and cavities, Flows in ducts and corners, Wake-vortex flows and bluff bodies

Linné Flow Centre
KTH Mechanics