Linear control of 3D disturbances on a flat-plate

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Abstract Using a number of localized sensors and actuators, a feedback controller is designed in order to reduce the growth of three-dimensional disturbances in the flat-plate boundary layer. A reduced-order model of the input-output system (composed of the linearized Navier–Stokes equations including inputs and outputs) is computed by projection onto a number of balanced truncation modes. It is shown that a model with 50 degrees of freedom captures the input-output behavior of the high-dimensional ($n \sim 10^7$) system. The controller is based on a classical LQG scheme with a row of three sensors in the spanwise direction connected to a row of three actuators further downstream. The controller minimizes the perturbation energy in a spatial region defined by a number of (objective) functions.

1 Input-output configuration

The three-dimensional input-output configuration considered is the extension of the two-dimensional case studied in [2, 3]. We focus on the dynamics and control of small amplitude perturbations about a steady base flow. The main numerical tool is a pseudo-spectral code that provides solutions of the linearized Navier-Stokes equations and its associated adjoint equations. For further details of the code, boundary conditions etc, we refer to [1, 2] . The computational domain has the dimensions $(L_x, L_y, L_z) = (500, 20, 160)$ and a resolution of $384 \times 81 \times 80$ grid points; the fringe region starts at $x = 400$. The Reynolds number is $Re = U_\infty \delta_0^*/\nu = 1000$, where δ_0^* is the inflow displacement thickness.

The plant (shown schematically in Fig. 1), written in an input-output form reads,

$$\dot{\mathbf{u}} = \mathbf{A}\mathbf{u} + \mathbf{B}_1 w + \mathbf{B}_2 u \quad (1)$$

$$z = \mathbf{C}_1 \mathbf{u} + l u \quad (2)$$

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$$v = \mathbf{C}_2\mathbf{u} + \alpha g \tag{3}$$

where $\mathbf{u}$ is the velocity field, $\mathbf{A} \in R^{n\times n}$ is the discretized and linearized Navier-Stokes equations, whereas the vector $\mathbf{B}_1 \in R^{n\times 1}$ and the matrix $\mathbf{B}_2 \in R^{n\times 3}$ provide the spatial distributions of the incoming disturbance upstream and the (three) actuators downstream. The output signals are extracted via the matrices $\mathbf{C}_1 \in R^{k\times n}$ and $\mathbf{C}_2 \in R^{3\times n}$ that define the spatial distributions of the sensors. The scalar quantities α and l are penalties of measurements noise $g(t)$ and control signal $u(t)$ (see [2]).

The system (1) is stable since all the eigenvalues of $\mathbf{A}$ are strictly to the left of the imaginary axis on the complex plane. However, the system is characterized by sensitive dynamics as it acts as an amplifier of disturbances. The upstream disturbance consist of the optimal localized initial condition computed by Monokrousos *et al.* [4], that provides the largest energy growth over a given time. For long time, the optimal initial condition is a three-dimensional wave-packet of Tollmien-Schilichting (TS) waves triggered by upstream tilted structures exploiting the Orr-mechanism. As the three actuators, we use localized volume forcing in the form of TS-like wave-packets.

The three sensors in the spanwise direction are also modelled as localized wave-packet structures. The k sensors $\mathbf{C}_1$, located further downstream are used to define the objective functional

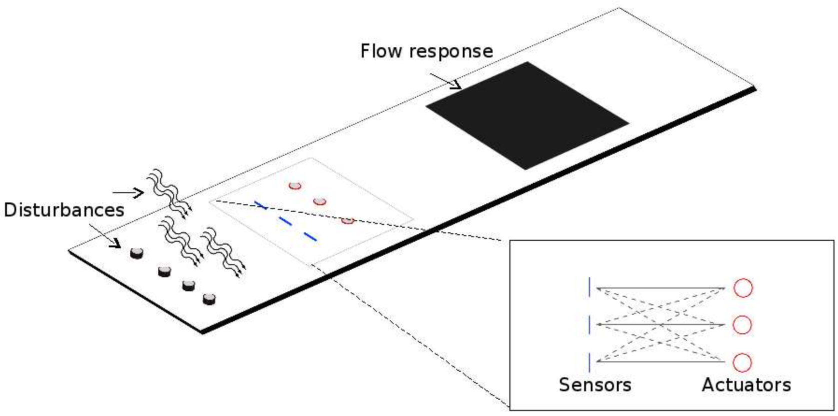

Fig. 1 Sketch of the input-output configuration considered. The disturbance ($\mathbf{B}_1$) is modelled as a localized TS-wavepacket located upstream at $(x_{\mathbf{B}_1}, y_{\mathbf{B}_1}, z_{\mathbf{B}_1}) = (20, 1, 0)$. A spanwise row ($\mathbf{C}_2$) of three sensors equally spaced along z ($\Delta_z = 40$) are used for estimation; the center sensor is located at $(x_{\mathbf{C}_2}, y_{\mathbf{C}_2}, z_{\mathbf{C}_2}) = (150, 1, 0)$. The actuator row ($\mathbf{B}_2$) has a similar configuration with the center actuator located at $(x_{\mathscr{B}_2}, y_{\mathscr{B}_2}, z_{\mathscr{B}_2}) = (200, 1, 0)$. A centralized controller is designed, i.e. all actuators are connected to sensors as shown by the inset figure. Further downstream a region, that is spanned by a number of basis functions ($\mathbf{C}_1$), is used to evaluate the disturbance dynamics and thus acts as an "objective function".

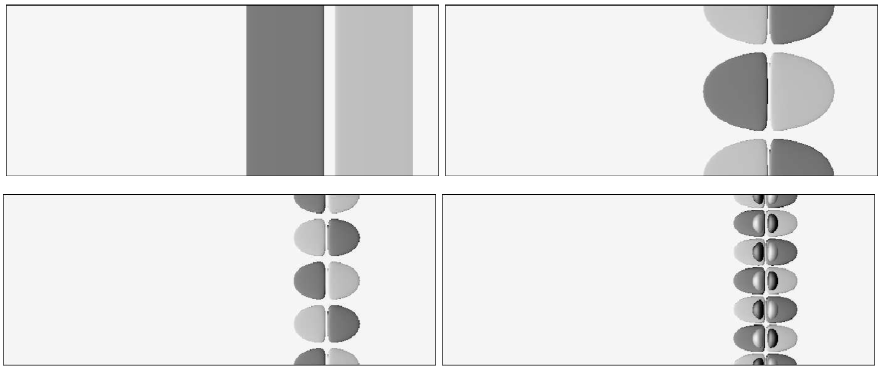

Fig. 2 Streamwise velocity component (positive is shown in black and negative in gray) of the basis functions $\mathbf{C}_{1,k}$ with $k = 1,\ldots,4$ given in equation (5) projected onto divergence-free subspace.

$$\|z\|^2 = \int_0^T |\mathbf{C}_1\mathbf{u}|^2 + l^2|u|^2 \, dt. \tag{4}$$

The aim of the controller is to determine the input signal $u(t)$, based on noisy sensor measurement $v(t)$ such that the above objective is minimized. Note that in this input-output framework, the controller minimizes the disturbance energy in a subspace of the domain, spanned by the basis $\{\mathbf{C}_{1,1},\ldots,\mathbf{C}_{1,k}\}$. One choice of basis (the so-called *output projection* [5]) are the POD modes obtained from the impulse response of all the inputs. This basis is empirical, i.e. it accurately represents the data used to generate it. Using output projection with few leading POD modes, the controlled system shows significantly smaller output signals $z(t)$ compared to the open-loop system. However, for three-dimensional disturbances, this does not correspond to an actual reduction of the total kinetic energy of the perturbation. Instead of including a very large number of POD modes, alternatively, a set of spanwise Fourier modes (see Fig. 2) localized in the streamwise and wall-normal directions can be used as to define a basis, of the form

$$\mathbf{C}_{1,k}\mathbf{u} = \int_\Omega \left(0, \exp\left(-(x-x_0)^2/\sigma_x^2 - y^2/\sigma_y^2, 0\right) \cos\left(\frac{2\pi(k-1)z}{N_z}\right)\mathbf{u} \, dxdydz. \tag{5}$$

Four modes – from $k = 1$ to $k = 4$ – localized around $x = 300$ are used in the present configuration.

2 Model reduction

In order to design a feedback controller, it is sufficient to capture the input-output (I/O) behavior of the system, rather than the entire perturbation dynamics. For a small number of inputs and outputs, the I/O behavior of a linear system can be

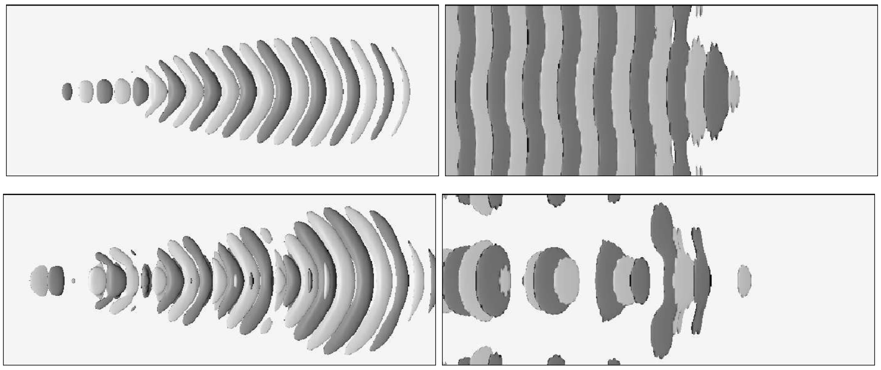

Fig. 3 Streamwise velocity (positive is shown in black and negative in gray) of the first (top row) and 10th (bottom row) balanced mode (left) and their associated adjoint mode (right).

described by a reduced-order model obtained via the balanced truncation method [6]. The reduced model retains the states affected easily by the inputs (controllable states) and the states that contribute the most to the outputs (observable states). Essentially, the method amounts to an oblique projection of the system (1), onto a number of so-called balanced modes which can be computed for high-dimensional plants using the *snapshots method* proposed in [5]: snapshots are collected from the impulse response of each input via a forward simulation, and of each output via a simulation of the adjoint system followed by one singular value decomposition (of the size of number of adjoint snapshots times forward snapshots).

Two balanced modes (first and 10th) and their associated adjoint modes are shown in Fig. 3. The first balanced mode is nearly two-dimensional and takes the shape of a TS wave-packet with a large amplitude downstream. This spatial structure is trigged with the least energy by the input $\mathbf{B}_1$. Its corresponding adjoint mode is essentially two-dimensional with its largest amplitude upstream. This structure, on the other hand, generates the largest response in the sensors $\mathbf{C}_1$. The higher balanced modes (bottom row in Fig. 3) look similar to the first mode, but are mainly characterized by different spatial wavelengths.

A reduced-order model of order 50 is found to capture the behavior between all the inputs and all the outputs of the Naiver-Stokes system of order 10^7. An example of the performance of the reduced-order model is shown in Fig. 4. With an impulse in $\mathbf{B}_1$, a (optimal) disturbance is introduced in the boundary-layer upstream that grows as it is convected in the downstream direction. The sensor outputs $z_1(t)$ and $z_2(t)$ extracted by the sensors $\mathbf{C}_{1,1}$ and $\mathbf{C}_{1,2}$ respectively, is shown with black solid lines. After an time-delay the sensors register a wave-packet; the signal eventually decays to zero as the disturbance leaves the computational box. In the same figure, the output signals computed using the reduced-order model is shown (circles) , where an impulse in the input of the reduced-order model ($\hat{B}_1$) results in the same response (extracted via the reduced sensors $\hat{C}_{1,1}$ and $\hat{C}_{1,2}$) as the full Navier-Stokes system, albeit the significant order reduction. The approximate Hankel sin-

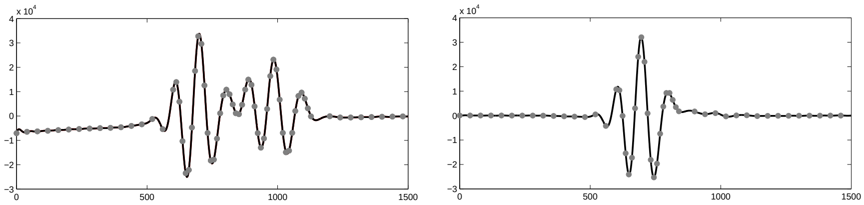

Fig. 4 Impulse response from $\mathbf{B}_1 \rightarrow \mathbf{C}_{1,1}$ (right) and $\mathbf{B}_1 \rightarrow \mathbf{C}_{1,2}$ (left) to a 3D TS wavepacket; the solid line represents the DNS ($n = 10^7$) and the dotted-line the reduced-model ($m = 50$).

gular values (not shown here) decay rapidely and the leading singular values come in pair similar to observations in previous studies [2]. A thorough analysis of the performance of the reduced-order and the model-reduction error will be presented elsewhere.

3 Controller design

The reduced-order model can be used to design a controller of low-order that will run “online”, next to the numerical experiments. Here, a classical Linear-Quadratic-Guassian (LQG) (see e.g. [7] for introduction in control theoretical tools from a fluid mechanics viewpoint) is designed, where all three sensors used for estimation ($\mathbf{C}_2$) are connected to all three actuators $\mathbf{B}_2$. Such a centralized controller minimizes the energy of the output signals (4) and more importantly the resulting closed-loop is guaranteed to be stable. A de-centralized controller – when the control signal of each actuator is based only on the output from the sensor located upstream and at the same spanwise location – was found both by RGA analysis [8] and by numerical experiments to result in an unstable closed-loop. This is partly due to the fact that localized disturbance introduced in the boundary-layer spreads (or widens) in the spanwise direction as it is convected downstream, resulting in a strong coupling in the spanwise direction.

The performance of the controller is shown in Fig. 5. The r.m.s. (streamwise velocity component integrated in spanwise and wall-normal directions and time) when forced upstream with temporal white noise is compared for three linearized DNS; in solid black the exponential growth of optimal TS wave-packet in the streamwise direction is observed; the dash-dotted and dashed lines show the disturbance development when the controller is active, i.e. when the measurements from the three sensors ($\mathbf{C}_2$) upstream are fed into a controller that provide the three actuators further downstream ($\mathbf{B}_2$) a control signal. The dashed line represents a “cheap” controller with $l = 10$, whereas the dashed-dotted line is “expensive” controller with $l = 100$ (for both controllers $\alpha = 0.1$). Near the location ($x = 200$) of the actuators the growth of perturbations is transformed into a decay; further downstream the perturbations again begin to growth, but their overall amplitude is reduced.

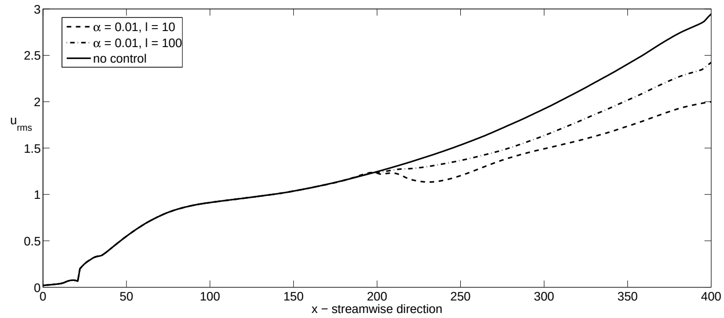

Fig. 5 R.M.S of the streamwise velocity component of the uncontrolled system (black line), the cheap controller (dashed) and expensive controller (dotted-dashed line).

In summary, a reduced-order model of order 50 is able to capture the input-output behavior between three-dimenstional disturbances, actuators, sensors and "objective functions". Using this model, efficient control strategies can be designed in order to damp the growth of small-amplitude perturbations inside the boundary-layer. A number of improvements are currently under investigation. The spatial structure of the sensors and actuators will be chosen in order to reflect what actually can be achieved in a practical experimental implementation (for instance with plasma actuators). Also, the choice of basis defining the objective functions in this study, was rather arbitrary and can be improved.

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