Spectral Representation of Oscillators

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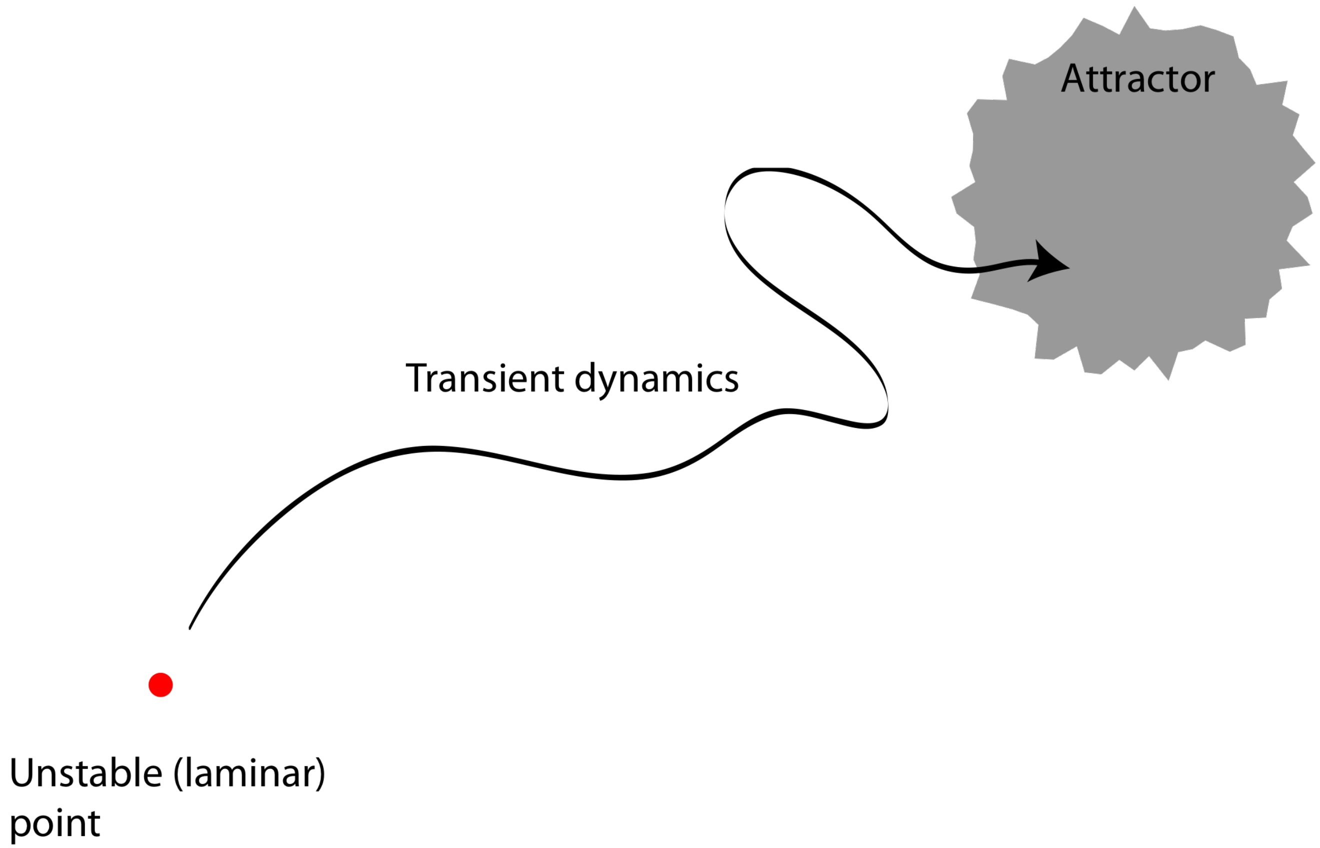
Supercritical Flow Dynamics



Spectral Expansion

- Expansion of flow field into
  - Operator approach: Koopman Operator

Koopman modes

$$\mathbf{u}(t) = \sum_{j=0}^{\infty} \mathbf{v}_j e^{\lambda_j t} \qquad \lambda_j \in \mathbb{C}$$

Koopman eigenvalues

  - Computational approach - Dynamic Mode Decomposition

$$\mathbf{u}(t) = \sum_{j=0}^{r-1} \tilde{\mathbf{v}}_j e^{\tilde{\lambda}_j t} \qquad \tilde{\lambda}_j \in \mathbb{C}$$

Ritz values

Ritz vectors

Mezic, 2005, Nonlin. Dyn.
Rowley etal 2009, JFM
Schmid, 2010, JFM

Observables

An observable $\mathbf{g}(\mathbf{u}) \in \mathcal{S}(\mathcal{M})$ with $\dim(\mathcal{S}) = \infty$ governed by

$$\frac{\partial \mathbf{g}}{\partial t} = (\mathbf{f}(\mathbf{u}) \cdot \nabla)\mathbf{g} = L\mathbf{g}.$$

Define Koopman operator

$$U_t = \exp(Lt)$$

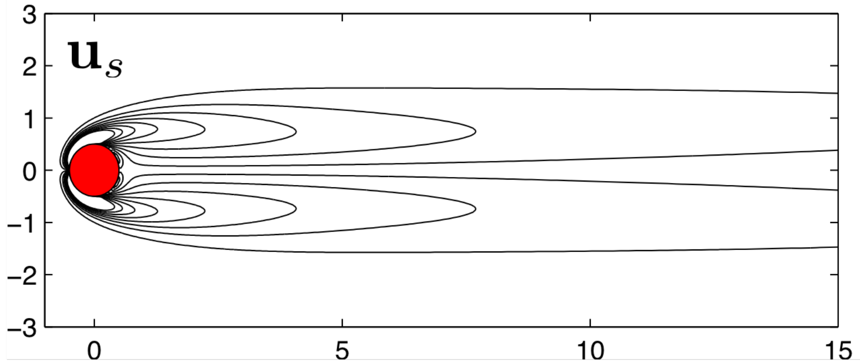
Spectral decomposition

$$\mathbf{g}(\mathbf{u}) = \sum_{j=0}^{\infty} \boldsymbol{\alpha}_j \phi_j(\mathbf{u}) e^{\lambda_j t} \quad = \sum_{j=0}^{\infty} \mathbf{v}_j e^{\lambda_j t}$$

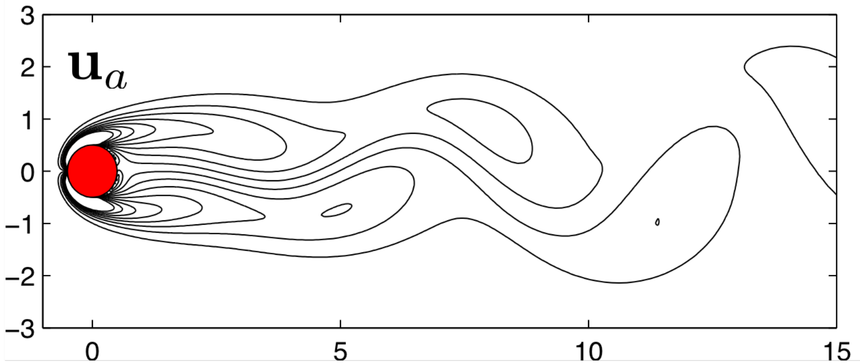
Mezic, Nonlin. Dyn. 2005
Mezic, Ann. Rev. Fluid Mech. 2013

Oscillator – Cylinder flow (Re=50)

- Navier-Stokes near the critical threshold for oscillation
  - Unstable equilibrium



  - **Attracting limit cycle**
    - Frequency ω
    - Growth rate $\sigma < 0$



Koopman Eigenvalues

– Formulas based on trace

$$\mathrm{Tr}\ U_t = \sum_{j=0}^{\infty} \exp(\lambda_j t).$$

or

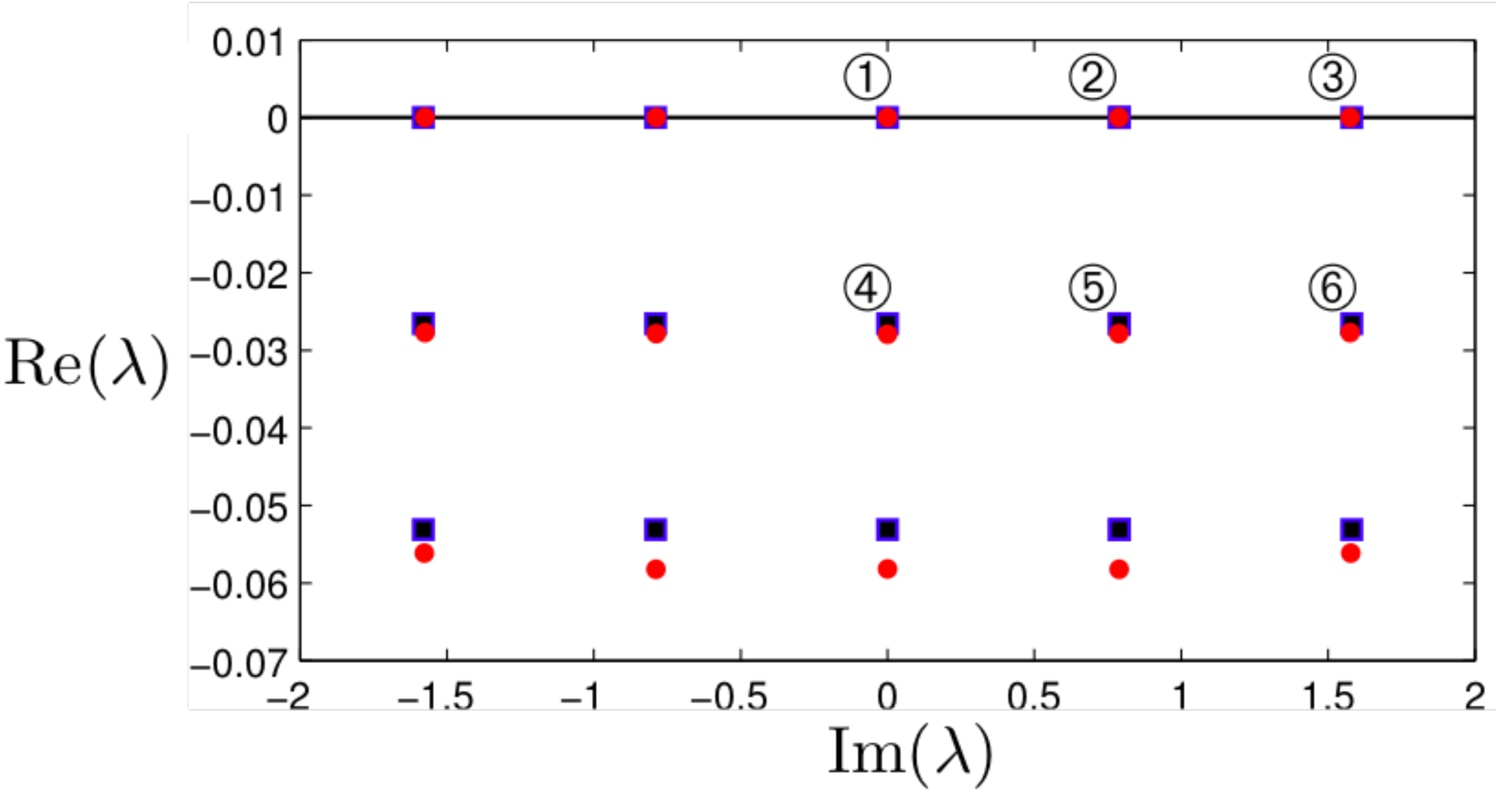
$$\mathrm{Tr}\ U_t = \int_{\mathcal{M}} \delta(\mathbf{u} - T_t(\mathbf{u}))d\mathbf{u}.$$

→ provides the Koopman eigenvalues

$$\lambda_{j,m} = j\sigma + im\omega, \qquad j = 0, 1, 2, \ldots, \quad m = 0, \pm 1, \pm 2, \ldots$$

Cvitanovic & Eckhardt 1991
Gaspard 1998

DMD of Attractor and Near Attractor



- Ritz values (DMD)
- Koopman eigenvalues

Koopman Modes

- Two steps
  1. Scale separation: fast limit cycle period but slow saturation

$$\frac{\partial A}{\partial \tau} = a_0 A - a_1 A|A|^2,$$

  2. Spectral expansion of S-L equation

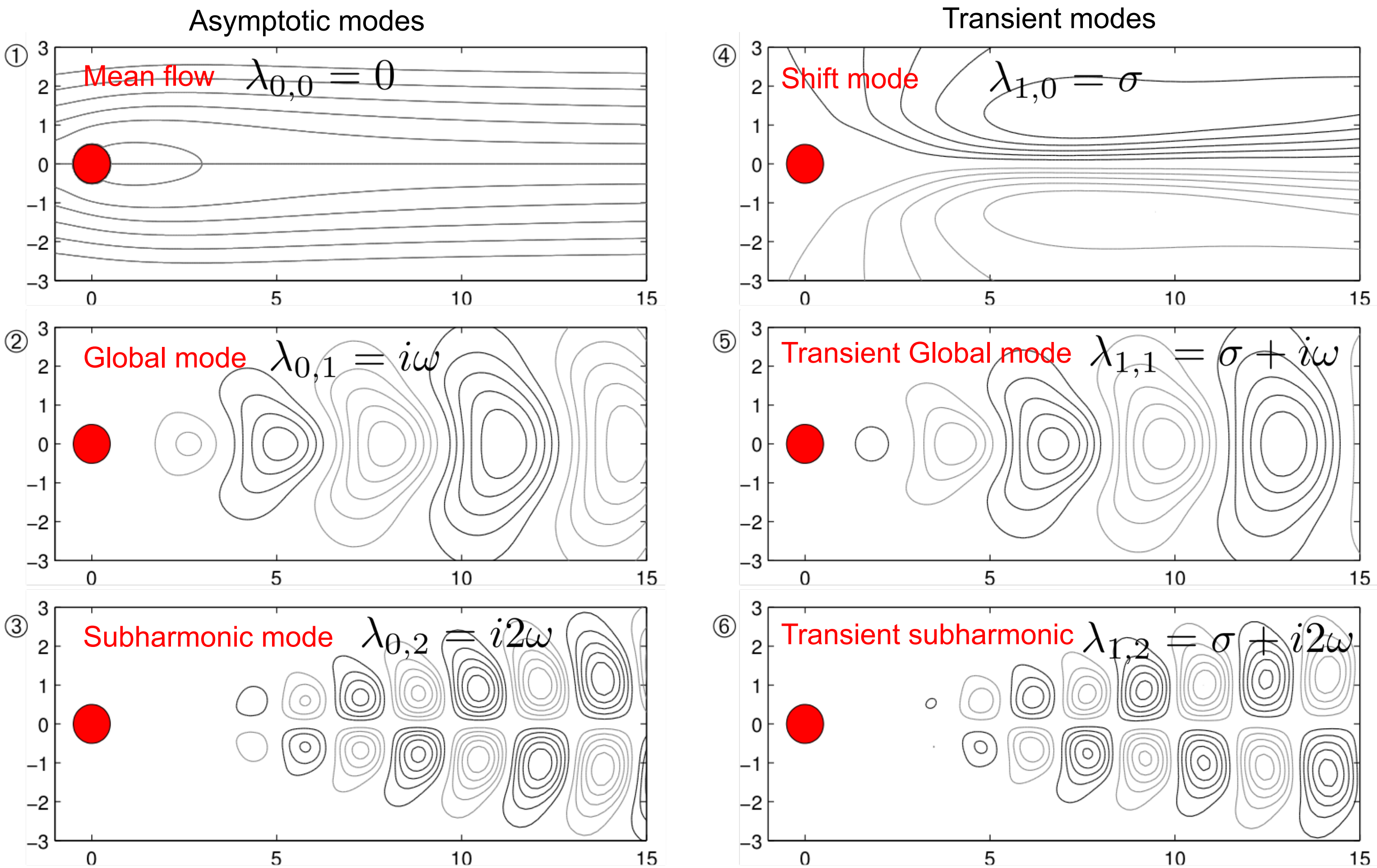
$$g(A) = \sum_j \alpha_j \phi_j \exp(\tilde{\lambda}_j \tau).$$

→ Provides Koopman Modes

Leading Koopman Modes

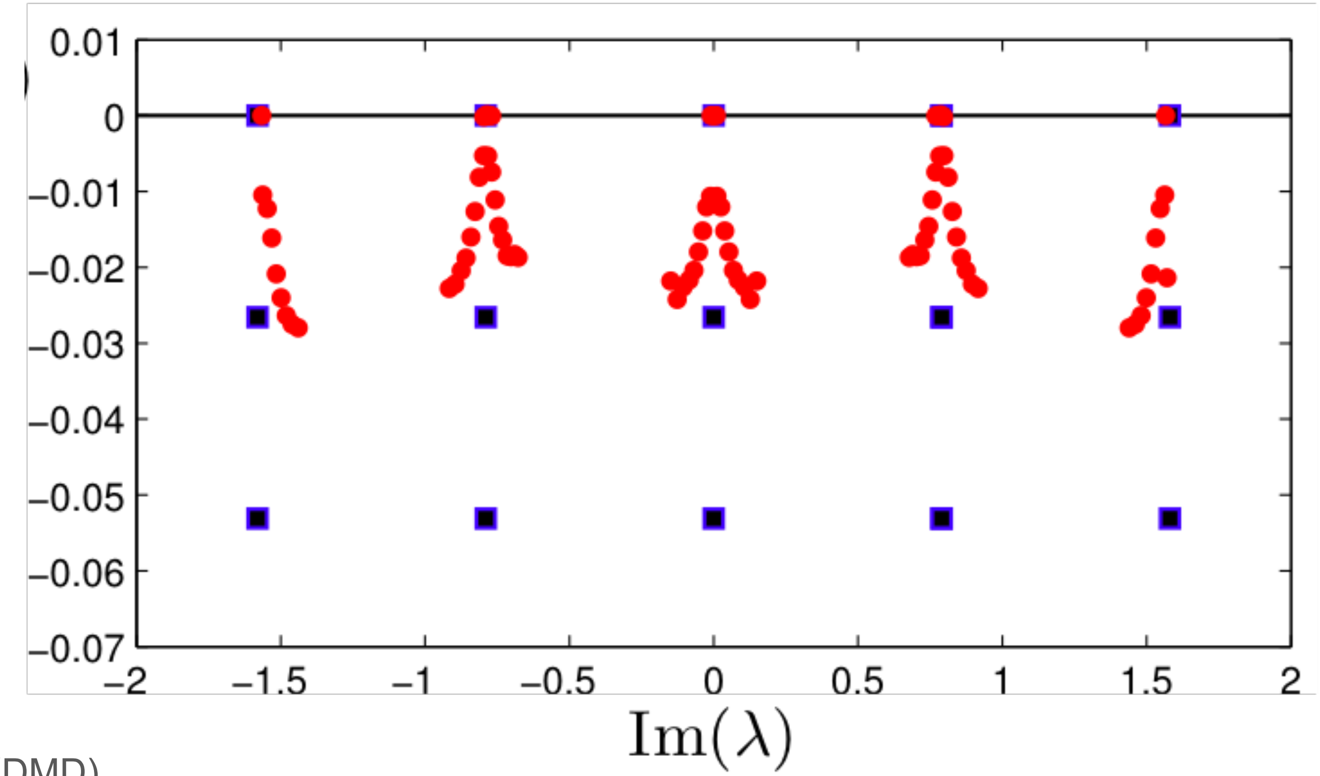
$\mathbf{v}_{0,0} = \mathbf{u}_s + \epsilon\tilde{\mathbf{u}}_2^{(1)} + \epsilon\mu\tilde{\mathbf{u}}_2^{(2)} + \ldots,$	$\lambda_{0,0} = 0$	Mean flow (asymptotic)
$\mathbf{v}_{1,0} = -\epsilon\mu\left(\frac{\mu}{r_0^2} - 1\right)\tilde{\mathbf{u}}_2^{(2)} + \ldots,$	$\lambda_{1,0} = \sigma$	Shift mode (transient)
$\mathbf{v}_{0,1} = \sqrt{\mu\epsilon}\left(\frac{\sqrt{\mu}}{r_0}\right)^{i\beta} e^{i\theta_0}\left(\tilde{\mathbf{u}}_1 + \epsilon\tilde{\mathbf{u}}_3^{(1)} + \epsilon\mu\tilde{\mathbf{u}}_3^{(2)} + \ldots\right),$	$\lambda_{0,1} = i\omega$	Global mode (asymptotic)
$\mathbf{v}_{1,1} = \frac{\sqrt{\mu\epsilon}}{2}\left(\frac{\sqrt{\mu}}{r_0}\right)^{i\beta}\left(\frac{\sqrt{\mu}}{r_0^2} - 1\right) e^{i\theta_0}\left((1 + i\beta)(\tilde{\mathbf{u}}_1 + \epsilon\tilde{\mathbf{u}}_3^{(1)}) + \epsilon\mu(3 + i\beta)\tilde{\mathbf{u}}_3^{(2)} + \ldots\right),$	$\lambda_{1,1} = \sigma + i\omega$	Global mode (asymptotic)
$\mathbf{v}_{0,2} = \epsilon\mu\left(\frac{\sqrt{\mu}}{r_0}\right)^{i2\beta} e^{i2\theta_0}\tilde{\mathbf{u}}_2^{(3)} + \ldots,$	$\lambda_{0,2} = i2\omega$	Subharmonic mode (asymptotic)
$\mathbf{v}_{1,2} = -\epsilon(1 + i\beta)\mu\left(\frac{\sqrt{\mu}}{r_0}\right)^{i2\beta} e^{i2\theta_0}\left(\frac{\sqrt{\mu}}{r_0^2} - 1\right)\tilde{\mathbf{u}}_2^{(3)}.$	$\lambda_{1,2} = \sigma + i2\omega$	Subharmonic mode (transient)

Ritz Vectors



DMD of Attractor and Near Attractor

Ritz cluster and form branches due to large algebraic growth



- Ritz values (DMD)
- Koopman eigenvalues

Conclusions

- Analytical results
  - Koopman eigenvalues form a lattice
  - Koopman modes correspond to mean flow, shift mode, global modes,..
- Computational results
  - Ritz vectors/values good approximation near and on attractor
  - Algebraic dynamics generates clusters/branches in spectrum