A method for extracting coherent structures from numerical & experimental data

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Koopman-Dynamic Modes

- Introduced 3 years ago
  - Dussin of papers in JFM/PoF...
  - Power grid systems, network systems, elasticity, etc...
  - An euphoria in the community...
A Free Water Jet Into a Pool

- Coherent structures
- Vortical structures
Cloud Formation

- Clouds roll up into Kelvin-Helmholtz vortices

- Two streams of different velocity: shear layer instabilities
Cloud Formation

- Karman vortex street developing behind an island
- Periodic vortex shedding

Island near Chilean coast
Motivation

– Can we extract periodic motion from flow?

– Where/how does unsteadiness arise? Can we describe transient motion of the flow?
Showcase: Jet in Crossflow

- Fluid injected through a hole into a crossflow

Smoke stacks  Volcano eruptions  Fuel injection/film cooling
The 4 Vortical Structures of Jet in Crossflow

- **Shear-layer vortices**
- **Horse-shoe/wall vortices**
- **Counter-rotating vortex pair**
- **Karman vortex street**

**Velocity ratio:**

\[ R = \frac{V}{U} = 3 \]

**Reynolds number:**

\[ Re = \frac{\delta^* U}{\nu} = 165 \]

*Kelso, Lim & Perry 1996, JFM*
DNS Movie

Ilak, Schlatter, Bagheri, & Henningson (2012 JFM, 2012 PoF)
Numerical Simulations

- Identified from DNS:
  - 2 events of vortex shedding
    (oscillation of separated region)

Bagheri, Schlatter, & Henningson (2009 JFM)
Spectral Expansion

- Expansion of flow field into

\[ u(t) = \sum_{j=0}^{\infty} v_j e^{\lambda_j t} \]

Koopman modes

\[ \lambda_j \in \mathbb{C} \]

Koopman eigenvalues

Rowley et al 2009, JFM

- Computational approach - Dynamic Mode Decomposition

Schmid, 2010, JFM
Koopman Spectrum of Jet in Crossflow

- Eigenvalues on the unit circle

\[ St = 0.017 \quad St = 0.141 \]

- Dominant frequencies match vortex shedding frequencies from DNS
- Computed using DMD (Dynamic Mode Decomposition) \((Schmid 2010)\)
Koopman Modes

• High-frequency mode:
  – Captures shear-layer structures
  – Matches first DNS-vortex shedding

\[ S_t = 0.141 \]

• Low-frequency mode
  – Captures wall structures
  – Matches second DNS-vortex shedding

\[ S_t = 0.017 \]

Rowley, Mezic, Bagheri, Schlatter & Henningson 2009, JFM
Extraction of Structures

- **Comparison:**
  - Linear Global modes: Normal mode analysis
  - Proper Orthogonal Decomposition (POD) modes
  - Koopman modes

<table>
<thead>
<tr>
<th>Mode</th>
<th>DNS</th>
<th>Global</th>
<th>POD</th>
<th>Koopman</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shear layer</td>
<td>0.141</td>
<td>0.169</td>
<td>0.138, 0.158, 0.121</td>
<td>0.141</td>
</tr>
<tr>
<td>Wall</td>
<td>0.017</td>
<td>0.043</td>
<td>0.0188, 0.0094, 0.158, 0.121</td>
<td>0.017</td>
</tr>
</tbody>
</table>
Koopman mode/DMD

- General formulation
- Koopman modes of oscillators
- Cylinder flow as prototype
Supercritical Flow Dynamics

Transient dynamics

Unstable (laminar) point

Attractor
Supercritical Flow Dynamics

- Focus on dynamics on and near the attractor

![Diagram showing focus on dynamics on and near the attractor with unstable (laminar) point and transient dynamics.]

Unstable (laminar) point

Neighborhood of attractor

Transient dynamics
Cylinder flow above Re=47

- Dynamical system has two critical elements
  - Unstable equilibrium (baseflow, fixed point, stationary solution,..)
  - Stable limit cycle (period orbit,..)

![Unstable equilibrium](image1)

![Periodic limit cycle](image2)
Governing Eqs of Fluid Motion

- Navier-Stokes equations

\[
\frac{\partial u_i}{\partial t} = -u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{Re} \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial p}{\partial x_j} + F_{b,i},
\]

\[
0 = \frac{\partial u_i}{\partial x_i},
\]

- Discretization \(\rightarrow\) dynamical system

- n-dimensional ODE: \(\dot{u}(t) = f(u; Re),\)

- Propagator: \(u(s + t) = T_t(u(s)).\)
Observables

- Any **function** of the flow field $g(u)$

- Infinitely many different observables: kinetic energy, probe, full flow field

- May have different dimensions and may be nonlinear
Governing Eqs for Observables

- Convection equation

\[ \frac{\partial g}{\partial t} = (f(u) \cdot \nabla)g = Lg. \]

  - Linear PDE (infinite-dimensional)

  - Hyperbolic system: observable “passive tracer” transported by \( f(u) \)

- Formal solution:

\[ g(u(t)) = \exp(Lt)g(u_0) = U_t g(u_0), \]
Spectrum of Koopman

- Linear operator $\rightarrow$ Spectrum

\[ L\phi_j(u) = \lambda_j \phi_j(u), \quad j = 0, 1, 2, \ldots \]

- Koopman eigenfunction $\rightarrow$ Koopman eigenvalue

- What do $\phi_j$ and $\lambda_j$ tell us about the flow?
Oscillator – Cylinder flow (Re=50)

- Attracting limit cycle
  - Poincare map
    \[ U_{k+1} = S^k U_1 \]
  - Linearized Poincare map
    \[ S^k_p (U_a + \delta U) = U_a + M \delta U + \ldots \]

- Frequency
  \[ \omega = \frac{t_p}{2\pi} \]

- Growth rate
  \[ \sigma < 0 \]
Koopman Eigenvalues

– Formulas based on trace

\[ \text{Tr } U_t = \sum_{j=0}^{\infty} \exp(\lambda_j t). \]

or

\[ \text{Tr } U_t = t_p \sum_{r=1}^{\infty} \frac{\delta(t - rt_p)}{\det(I - Mr)}. \]

→ Provides a subset of the Koopman eigenvalues

\[ \lambda_{j,m} = j\sigma + im\omega, \quad j = 0, 1, 2, \ldots, \]
\[ m = 0, \pm 1, \pm 2, \ldots \]
Analytical & Computational spectrum

Bagheri, 2012, Submitted JFM
Analytical & Computational spectrum

Bagheri, 2012, Submitted JFM
Analytical & Computational spectrum

Re(\(\lambda\))
Growth rate

Im(\(\lambda\))
Frequency

- Ritz values (DMD)
- Koopman eigenvalues

Bagheri, 2012, Submitted JFM
Analytical & Computational spectrum

Re(\(\lambda\))  
Growth rate

Im(\(\lambda\))  
Frequency

Ritz values (DMD)  
Koopman eigenvalues

Steady structures

Bagheri, 2012, Submitted JFM
Analytical & Computational spectrum

Structures oscillating with fundamental frequency

Re(\lambda)
Growth rate

Im(\lambda)
Frequency

Ritz values (DMD)
Koopman eigenvalues

Bagheri, 2012, Submitted JFM
Analytical & Computational spectrum

Structures oscillating with subharmonic frequency

$\text{Re}(\lambda)$
Growth rate

$\text{Im}(\lambda)$
Frequency

- Ritz values (DMD)
- Koopman eigenvalues

Bagheri, 2012, Submitted JFM
Koopman modes

Expansion of an observable in Koopman eigenfunctions

\[ g(u) = \sum_{j=0}^{\infty} \alpha_j \phi_j(u) e^{\lambda_j t} = \sum_{j=0}^{\infty} v_j e^{\lambda_j t} \]

\[ \lambda_j \in \mathbb{C} \quad \text{Koopman eigenvalues} \]

Mezic, Nonlin. Dyn. 2005
Koopman Modes

- Two steps
  1. Scale separation: fast limit cycle period but slow saturation

\[ \frac{\partial A}{\partial \tau} = a_0 A - a_1 A|A|^2, \]

2. Spectral expansion of S-L equation

\[ g(A) = \sum_j \alpha_j \phi_j \exp(\tilde{\lambda}_j \tau). \]

→ Provides Koopman Modes

Bagheri, 2012, Submitted JFM
Koopman modes (analytical)

\begin{align*}
\v_0,0 &= \mathbf{u}_s + \epsilon \tilde{\mathbf{u}}_{2}^{(1)} + \epsilon \mu \tilde{\mathbf{u}}_{2}^{(2)} + \ldots, \\
\v_{1,0} &= -\epsilon \mu \left( \frac{\mu}{r_0^2} - 1 \right) \tilde{\mathbf{u}}_{2}^{(2)} + \ldots, \\
\v_{0,1} &= \sqrt{\mu} \epsilon \left( \frac{\sqrt{\mu}}{r_0} \right)^{i\beta} e^{i\theta_0} \left( \tilde{\mathbf{u}}_1 + \epsilon \tilde{\mathbf{u}}_{3}^{(1)} + \epsilon \mu \tilde{\mathbf{u}}_{3}^{(2)} + \ldots \right), \\
\v_{1,1} &= \frac{\sqrt{\mu} \epsilon}{2} \left( \frac{\sqrt{\mu}}{r_0^2} \right)^{i\beta} \left( \frac{\sqrt{\mu}}{r_0^2} - 1 \right) e^{i\theta_0} \left( 1 + i\beta \right) \left( \tilde{\mathbf{u}}_1 + \epsilon \tilde{\mathbf{u}}_{3}^{(1)} \right) \\
&\quad + \epsilon \mu (3 + i\beta) \tilde{\mathbf{u}}_{3}^{(2)} + \ldots, \\
\v_{0,2} &= \epsilon \mu \left( \frac{\sqrt{\mu}}{r_0} \right)^{i2\beta} e^{i2\theta_0} \tilde{\mathbf{u}}_{2}^{(3)} + \ldots, \\
\v_{1,2} &= -\epsilon (1 + i\beta) \mu \left( \frac{\sqrt{\mu}}{r_0} \right)^{i2\beta} e^{i2\theta_0} \left( \frac{\sqrt{\mu}}{r_0^2} - 1 \right) \tilde{\mathbf{u}}_{2}^{(3)}.
\end{align*}

\begin{align*}
\lambda_{0,0} &= 0 & \text{Mean flow (asymptotic)} \\
\lambda_{1,0} &= \sigma & \text{Shift mode (transient)} \\
\lambda_{0,1} &= i\omega & \text{Global mode (asymptotic)} \\
\lambda_{1,1} &= \sigma + i\omega & \text{Global mode (asymptotic)} \\
\lambda_{0,2} &= i2\omega & \text{Subharmonic mode (asymptotic)} \\
\lambda_{1,2} &= \sigma + i2\omega & \text{Subharmonic mode (transient)}
\end{align*}

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Spectral Expansion

- Expansion of flow field into
  - Operator approach: Koopman Operator

\[
    u(t) = \sum_{j=0}^{\infty} v_j e^{\lambda_j t}
\]

Koopman modes

\[
    \lambda_j \in \mathbb{C}
\]

Koopman eigenvalues

- Computational approach - Dynamic Mode Decomposition

\[
    u(t) = \sum_{j=0}^{r-1} \tilde{v}_j e^{\tilde{\lambda}_j t}
\]

Ritz vectors

\[
    \tilde{\lambda}_j \in \mathbb{C}
\]

Ritz values

Rowley et al 2009, JFM
Schmid, 2010, JFM
Koopman modes (computational)

Asymptotic modes

1. Mean flow $\lambda_{0,0} = 0$

2. Global mode $\lambda_{0,1} = i\omega$

3. Subharmonic mode $\lambda_{0,2} = i2\omega$

Transient modes

4. Shift mode $\lambda_{1,0} = \sigma$

5. Transient Global mode $\lambda_{1,1} = \sigma + i\omega$

6. Transient subharmonic $\lambda_{1,2} = \sigma + i2\omega$

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Koopman mode: Mean Flow

Mean flow $\lambda_{0,0} = 0$
Koopman mode: Global mode

Global mode \( \lambda_{0,1} = i\omega \)
Koopman mode: Subharmonic Global mode

Subharmonic Global mode $\lambda_{0,2} = i2\omega$
Koopman mode: Shift mode

\[ \lambda_{1,0} = \sigma \]
Conclusions

• Analytical results
  – Koopman eigenvalues form a lattice (trace formula)
  – Koopman modes correspond to mean flow, shift mode, global modes,..

• Computational results
  – Ritz vectors/values good approximation near and on attractor
  – Algebraic dynamics generates clusters/branches in spectrum