A method for extracting coherent structures from numerical & experimental data

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Koopman-Dynamic Modes

- Introduced 3 years ago
  - Dussin of papers in JFM/PoF…
  - Power grid systems, network systems, elasticity, etc…
  - An euphoria in the community…

REVIEWS IN ADVANCE

Review in Advance first posted online on October 5, 2012. (Changes may still occur before final publication online and in print.)

Analysis of Fluid Flows via Spectral Properties of the Koopman Operator

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Keywords

Koopman mode expansion, dynamic mode decomposition, global modes, Arnoldi algorithm

Abstract

This article reviews theory and applications of Koopman modes in fluid mechanics. Koopman mode decomposition is based on the surprising fact, discovered in Mezić (2005), that normal modes of linear oscillations have their natural analogs—Koopman modes—in the context of nonlinear dynamics. To pursue this analogy, one must change the representation of the system from the state-space representation to the dynamics governed by the linear Koopman operator on an infinite-dimensional space of observables. Whereas Koopman in his original paper dealt only with measure-preserving transformations, the discussion here is predominantly on dissipative systems arising from Navier-Stokes evolution. The analysis is based on spectral properties of the Koopman operator. Aspects of point and continuous parts of the spectrum are discussed. The point spectrum corresponds to isolated frequencies of oscillation present in the fluid flow, and also to growth rates of stable and unstable modes. The continuous part of the spectrum corresponds to chaotic motion on the attractor. A method of computation of the spectrum and the associated Koopman modes is discussed in terms of generalized Laplace analysis. When applied to a generic observable, this method uncovers the full point spectrum. A computational alternative is given by Arnoldi-type methods, leading to so-called dynamic mode decomposition, and I discuss the connection and differences between these two methods. A number of applications are reviewed in which decompositions of this type have been pursued. Koopman mode theory unifies and provides a rigorous background for a number of different concepts that have been advanced in fluid mechanics, including global mode analysis, triple decomposition, and dynamic mode decomposition.

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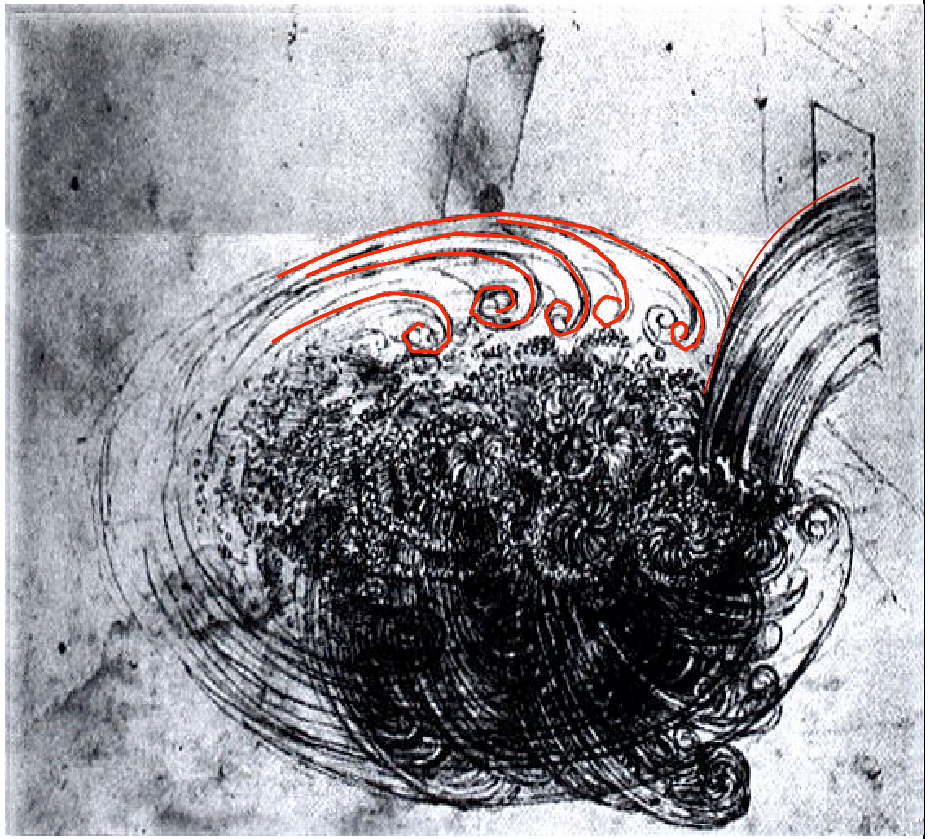
A Free Water Jet Into a Pool

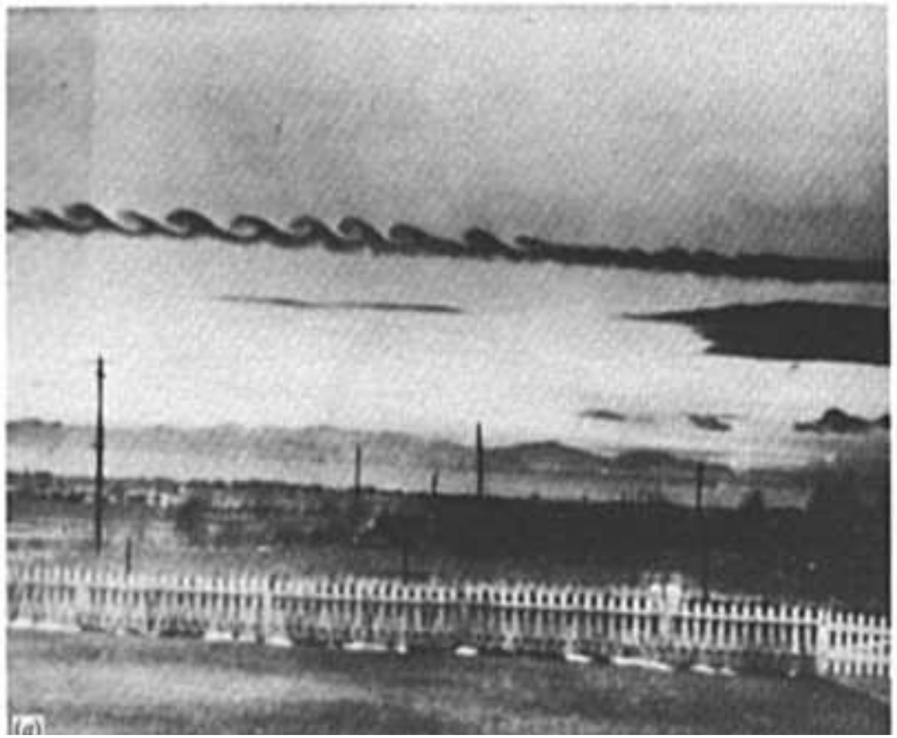
- Coherent structures
- Vortical structures

Leonardo da Vinci



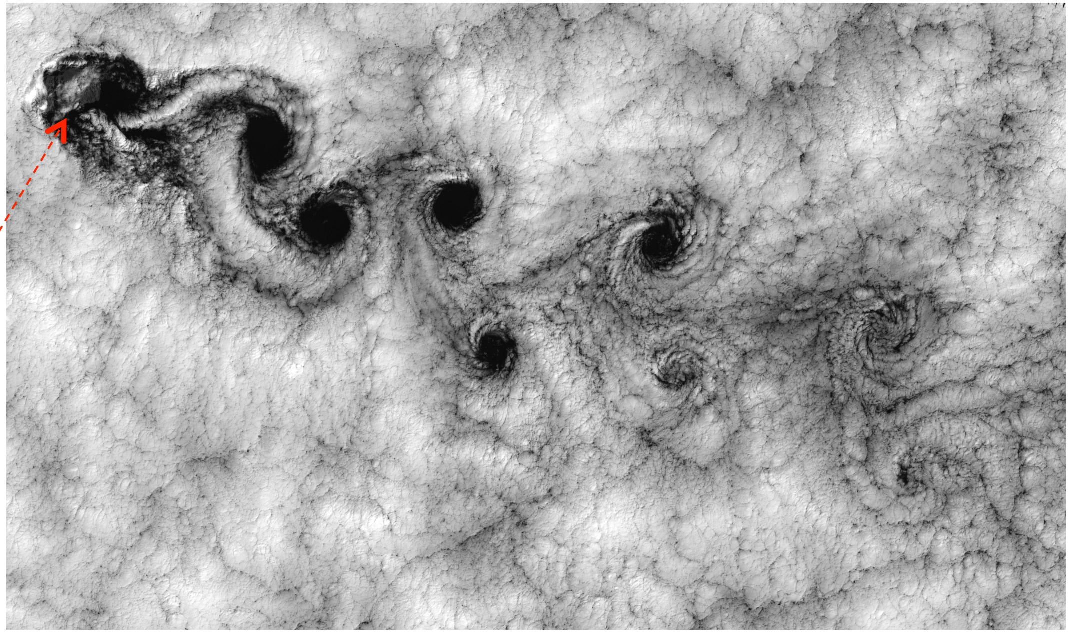
Cloud Formation

- Clouds roll up into Kelvin-Helmholtz vortices
- Two streams of different velocity: shear layer instabilities



Clould Formation

- Karman vortex street developing behind an island
- Periodic vortex shedding



Island near Chilean coast

Motivation

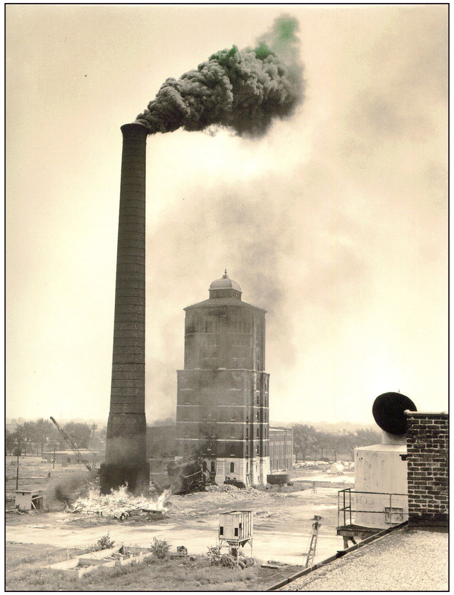
– Can we extract periodic motion from flow?

– Where/how does unsteadiness arise? Can we describe transient motion of the flow?

Showcase: Jet in Crossflow

- Fluid injected through a hole into a crossflow

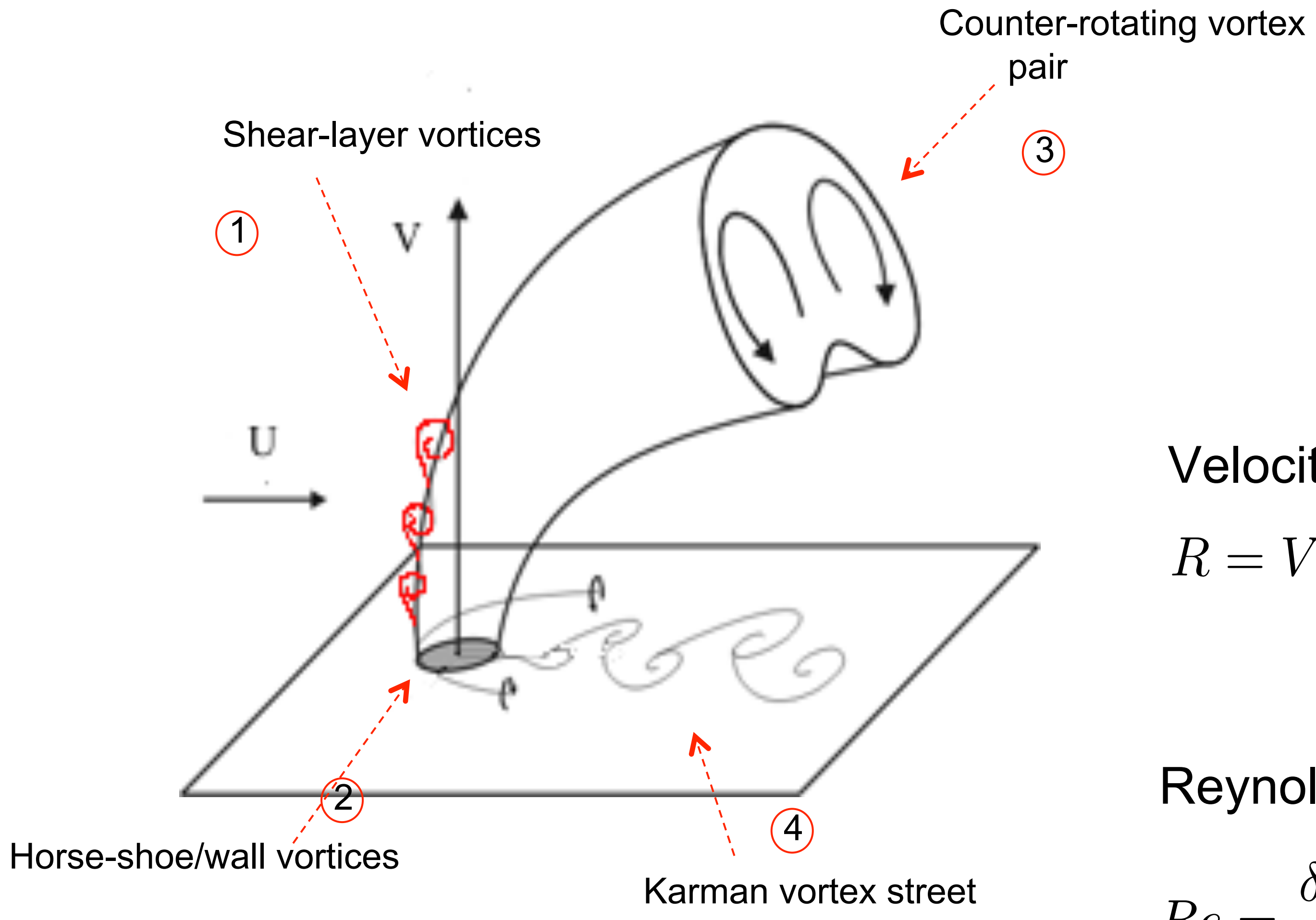
Smoke stacks



Volcano eruptions

Fuel injection/film cooling

The 4 Vortical Structures of Jet in Crossflow



Kelso, Lim & Perry 1996, JFM

Velocity ratio:

$$R = V/U = 3$$

Reynolds number:

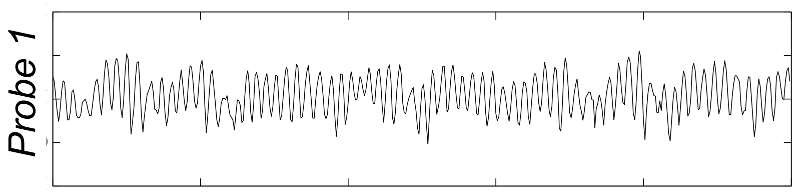
$$Re = \frac{\delta_0^* U}{\nu} = 165$$

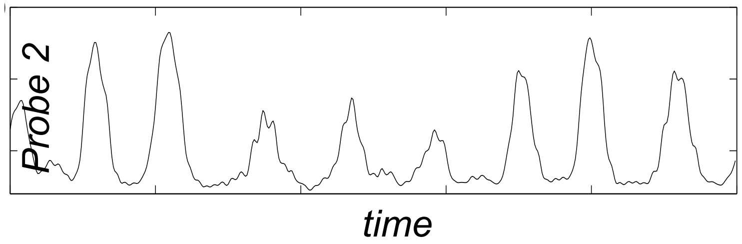
DNS Movie

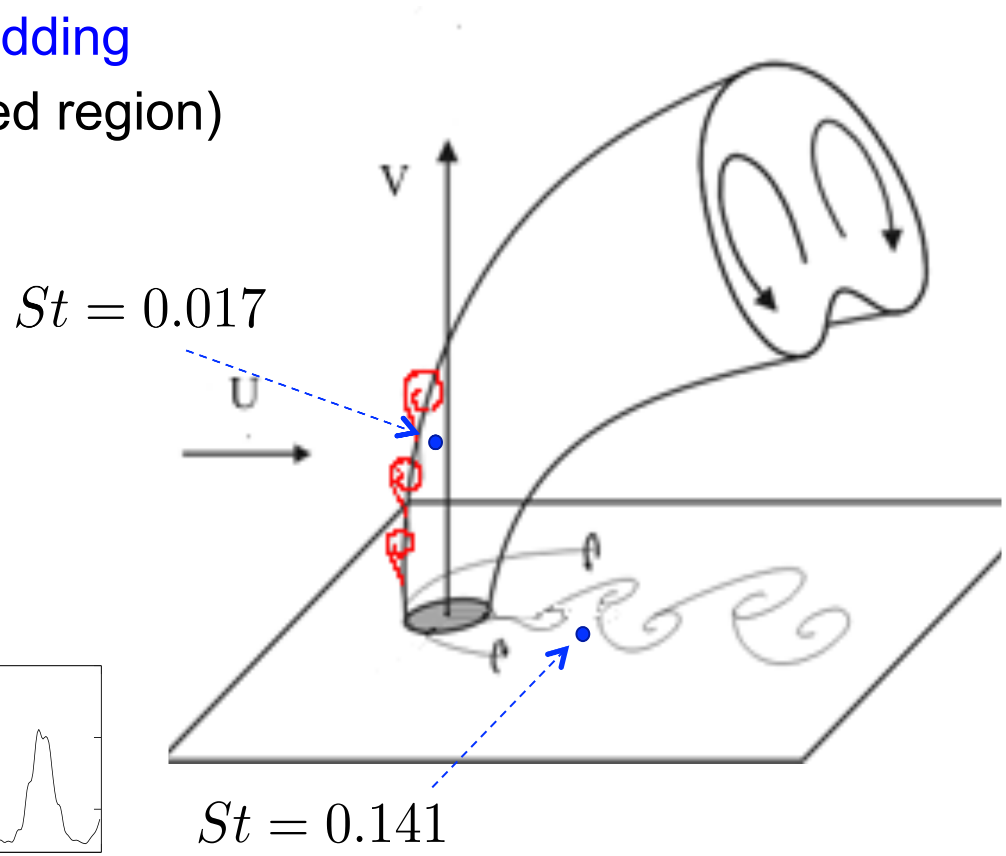
Ilak, Schlatter, Bagheri, & Henningson (2012 JFM, 2012 PoF)

Numerical Simulations

- Identified from DNS:
  - 2 events of vortex shedding (oscillation of separated region)







Bagheri, Schlatter, & Henningson (2009 JFM)

Spectral Expansion

- Expansion of flow field into

Koopman modes

$$\mathbf{u}(t) = \sum_{j=0}^{\infty} \mathbf{v}_j e^{\lambda_j t} \qquad \lambda_j \in \mathbb{C}$$

Koopman eigenvalues

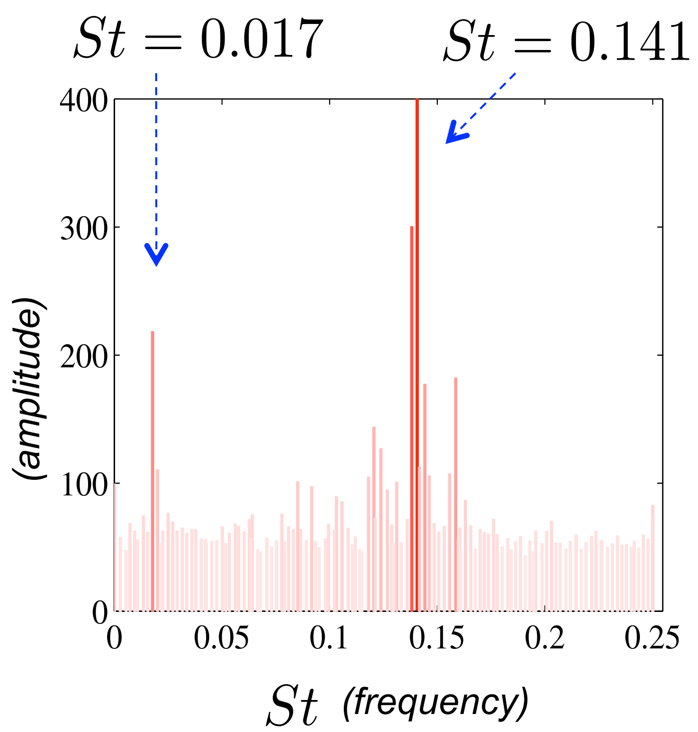
Mezic, 2005, Nonlin. Dyn.

Rowley et al 2009, JFM

  – Computational approach - Dynamic Mode Decomposition
    Schmid, 2010, JFM

Koopman Spectrum of Jet in Crossflow

- Eigenvalues on the unit circle



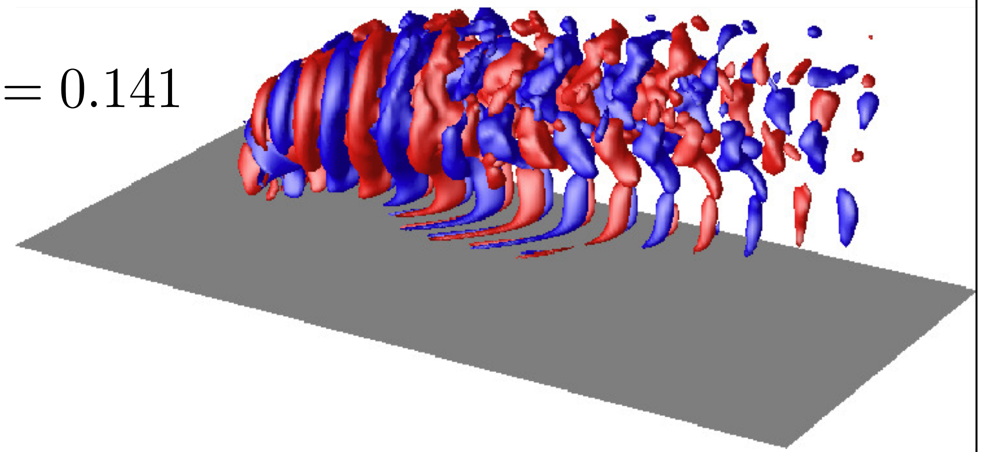
  - Dominant frequencies match vortex shedding frequencies from DNS
  - Computed using DMD (Dynamic Mode Decomposition) *(Schmid 2010)*

Koopman Modes

Positive streamwise velocity
Negative streamwise velocity

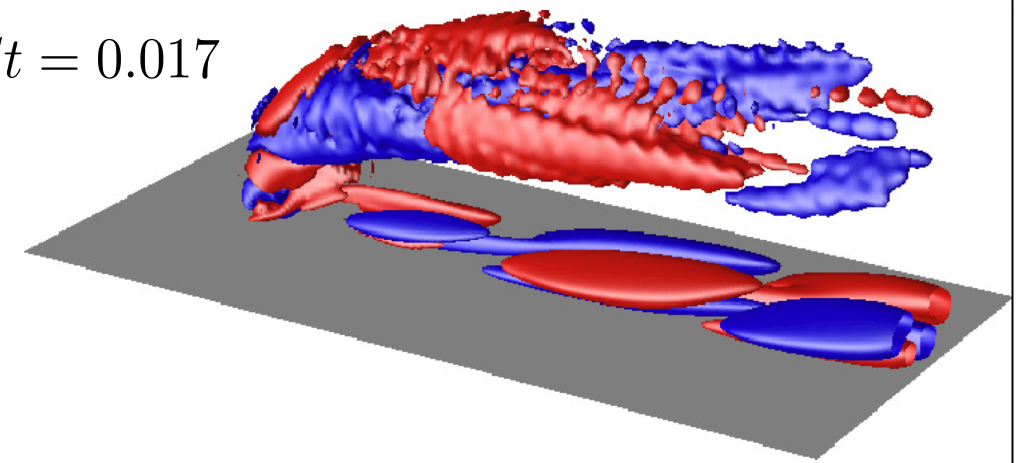
- High-frequency mode:
  - Captures shear-layer structures
  - Matches first DNS-vortex shedding

$St = 0.141$



- Low-frequency mode
  - Captures wall structures
  - Matches second DNS-vortex shedding

$St = 0.017$



Rowley, Mezic, Bagheri, Schlatter & Henningson 2009, JFM

Extraction of Structures

- Comparison:
  - Linear Global modes: Normal mode analysis
  - Proper Orthogonal Decomposition (POD) modes
  - Koopman modes

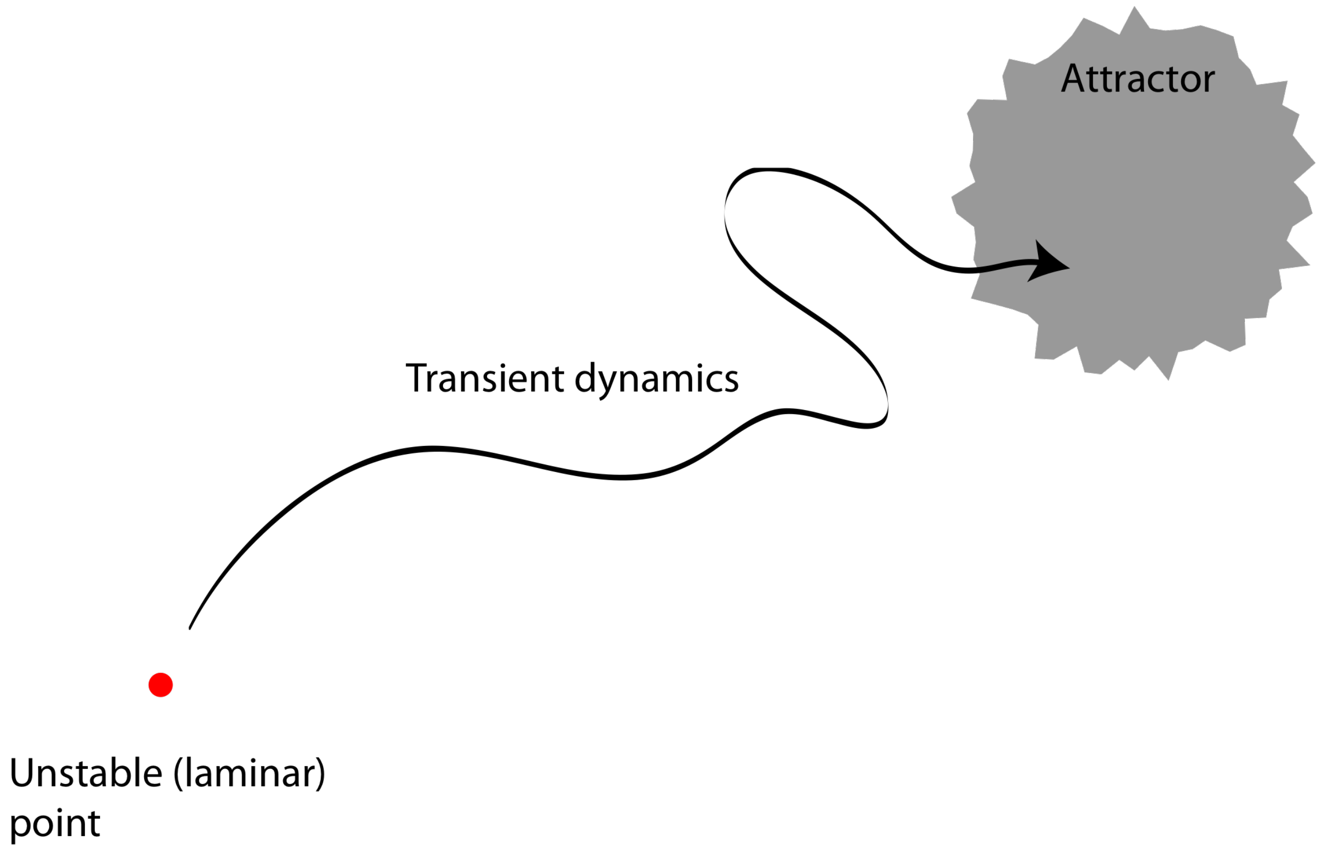
Mode	DNS	Global	POD	Koopman
Shear layer	0.141	0.169	0.138, 0.158, 0.121	0.141
Wall	0.017	0.043	0.0188, 0.0094, 0.158, 0.121	0.017

Bagheri (2012 ARCME)

Koopman mode/DMD

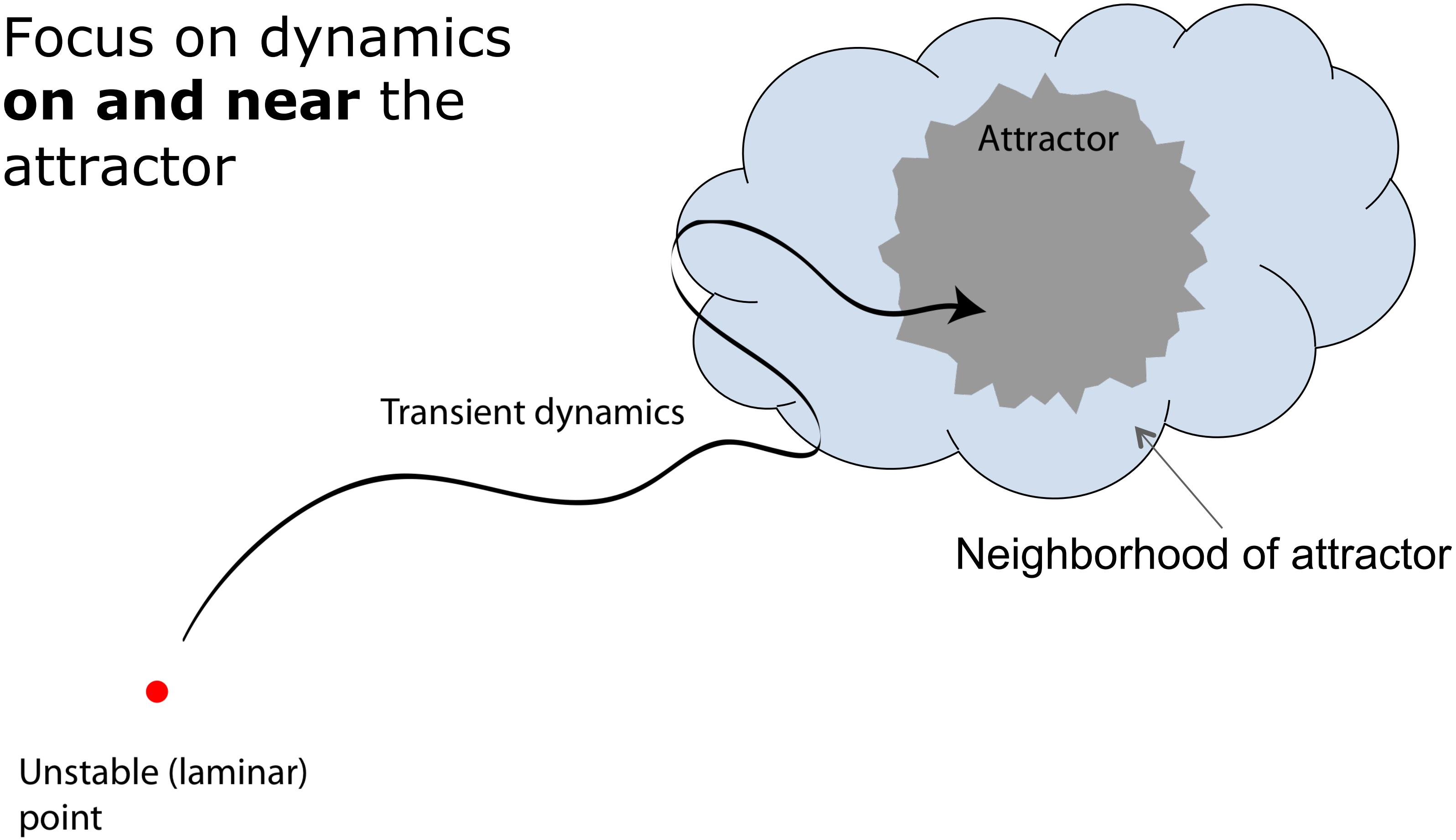
- General formulation
- Koopman modes of oscillators
- Cylinder flow as prototype

Supercritical Flow Dynamics



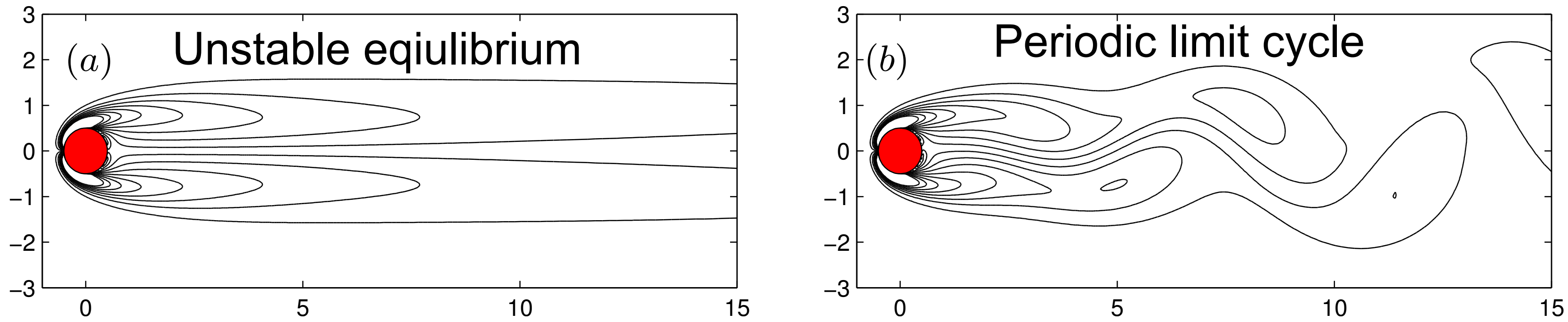
Supercritical Flow Dynamics

- Focus on dynamics **on and near** the attractor



Cylinder flow above Re=47

- Dynamical system has two critical elements
  - Unstable equilibrium (baseflow, fixed point, stationary solution,..)
  - Stable limit cycle (period orbit,…)



Governing Eqs of Fluid Motion

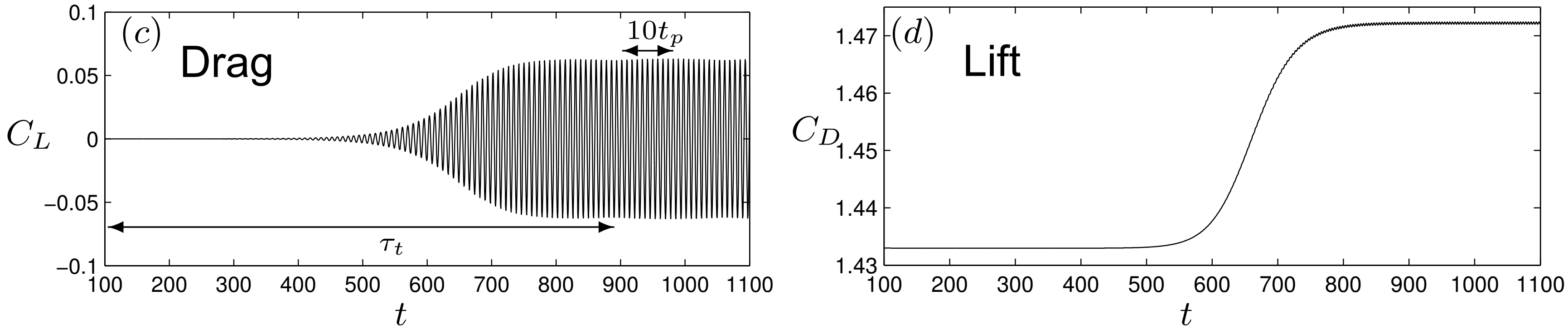
- Navier-Stokes equations

$$
\begin{aligned}
\frac{\partial u_i}{\partial t} &= -u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{Re} \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial p}{\partial x_j} + F_{b,i}, \\
0 &= \frac{\partial u_i}{\partial x_i},
\end{aligned}
$$

- Discretization → dynamical system
  - n-dimensional ODE: $\dot{\mathbf{u}}(t) = \mathbf{f}(\mathbf{u}; Re),$
  - Propagator: $\mathbf{u}(s+t) = T_t(\mathbf{u}(s)).$

Observables

- Any ***function*** of the flow field $\mathbf{g}(\mathbf{u})$



  – Infinitely many different observables: kinetic energy, probe, full flow field

  – May have different dimensions and may be nonlinear

Governing Eqs for Observables

- Convection equation

$$\frac{\partial \mathbf{g}}{\partial t} = (\mathbf{f}(\mathbf{u}) \cdot \nabla)\mathbf{g} = L\mathbf{g}.$$

  - Linear PDE (infinite-dimensional)
  - Hyperbolic system: observable “passive tracer” transported by $\mathbf{f}(\mathbf{u})$

- Formal solution:

Koopman Operator

$$\mathbf{g}(\mathbf{u}(t)) = \exp(Lt)\mathbf{g}(\mathbf{u}_0) = U_t\mathbf{g}(\mathbf{u}_0),$$

Spectrum of Koopman

- Linear operator → Spectrum

$$L\phi_j(\mathbf{u}) = \lambda_j \phi_j(\mathbf{u}), \qquad j = 0, 1, 2, \ldots$$

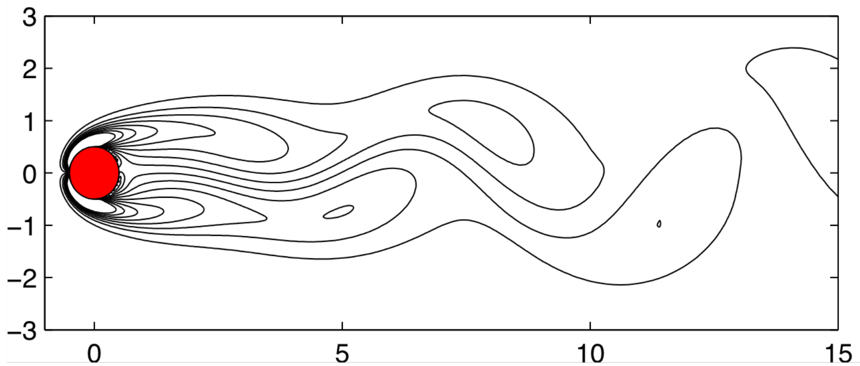
Koopman eigenfunction

Koopman eigenvalue

- What do ϕ_j and λ_j tell us about the flow?

Oscillator – Cylinder flow (Re=50)

- Attracting limit cycle



  – Poincare map

$$\mathbf{U}_{k+1} = S^k \mathbf{U}_1$$

  – Linearized Poincare map

$$S^{k_p}(\mathbf{U}_a + \delta\mathbf{U}) = \mathbf{U}_a + \mathbf{M}\delta\mathbf{U} + \dots$$

Monodromy/Floquet matrix

- Frequency $\omega = \dfrac{t_p}{2\pi},$
- Growth rate $\sigma < 0$

Koopman Eigenvalues

– Formulas based on trace

$$\mathrm{Tr}\ U_t = \sum_{j=0}^{\infty} \exp(\lambda_j t).$$

or

$$\boxed{\mathrm{Tr}\ U_t = t_p \sum_{r=1}^{\infty} \frac{\delta(t - r t_p)}{|\det(\mathbf{I} - \mathbf{M}^r)|}.}$$

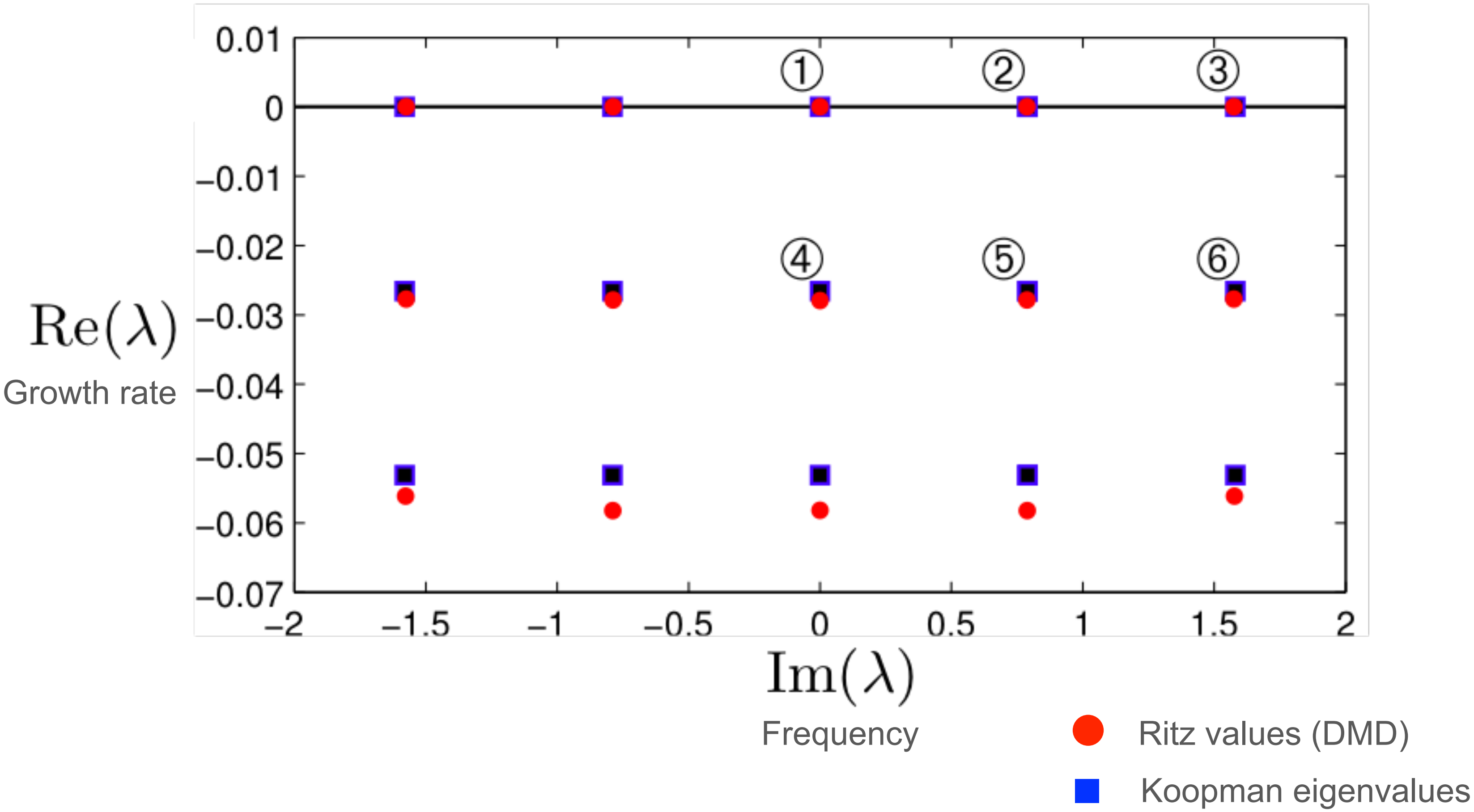
Trace Formula

→ Provides a subet of the Koopman eigenvalues

$$\lambda_{j,m} = j\sigma + im\omega, \qquad \begin{array}{l} j = 0, 1, 2, \ldots, \\ m = 0, \pm 1, \pm 2, \ldots \end{array}$$

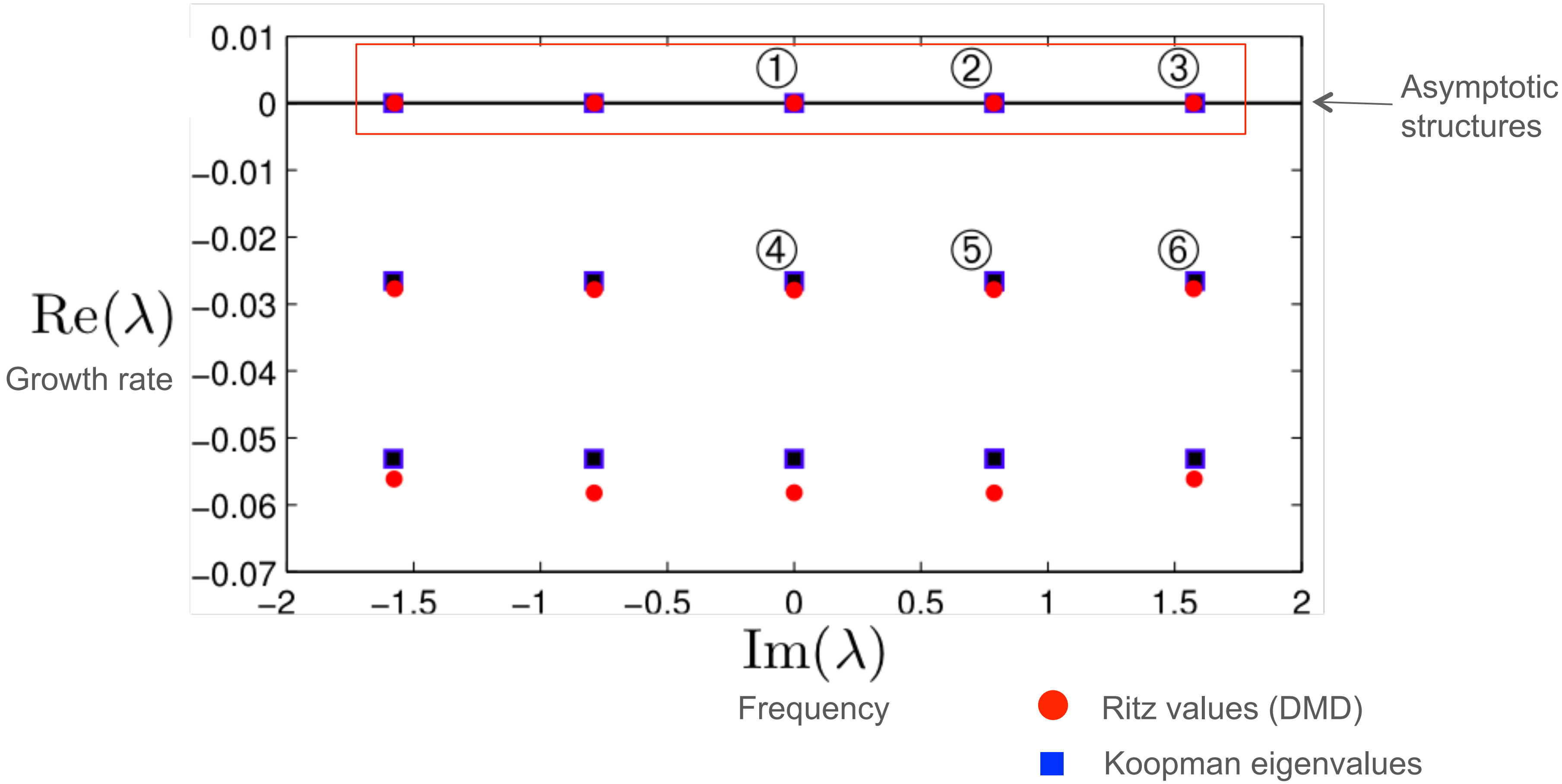
Cvitanovic & Eckhardt, 1991, Phys. A. Math
Gaspard 1998 (Cambridge)

Analytical & Computational spectrum



Bagheri, 2012, Submitted JFM

Analytical & Computational spectrum



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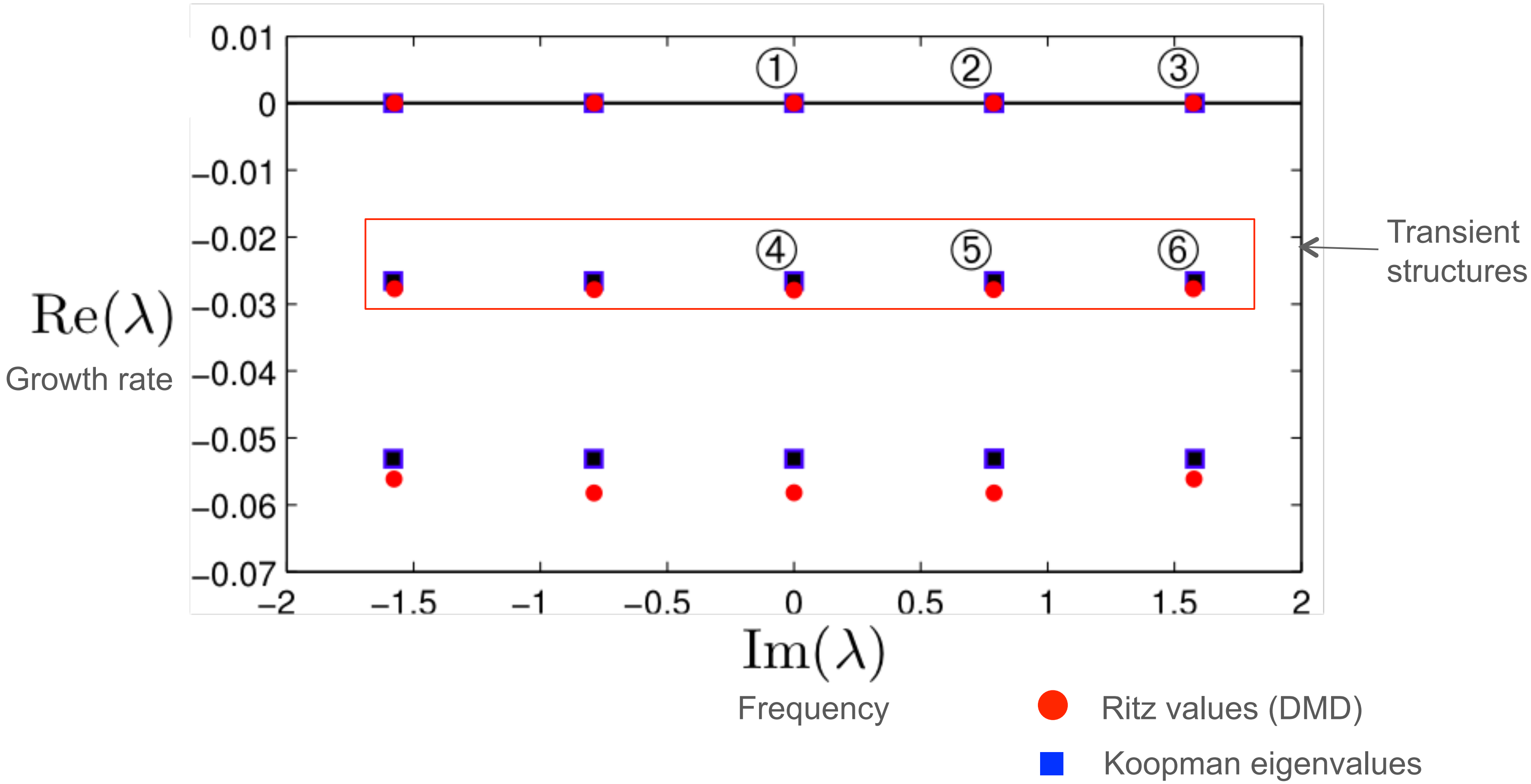
Analytical & Computational spectrum



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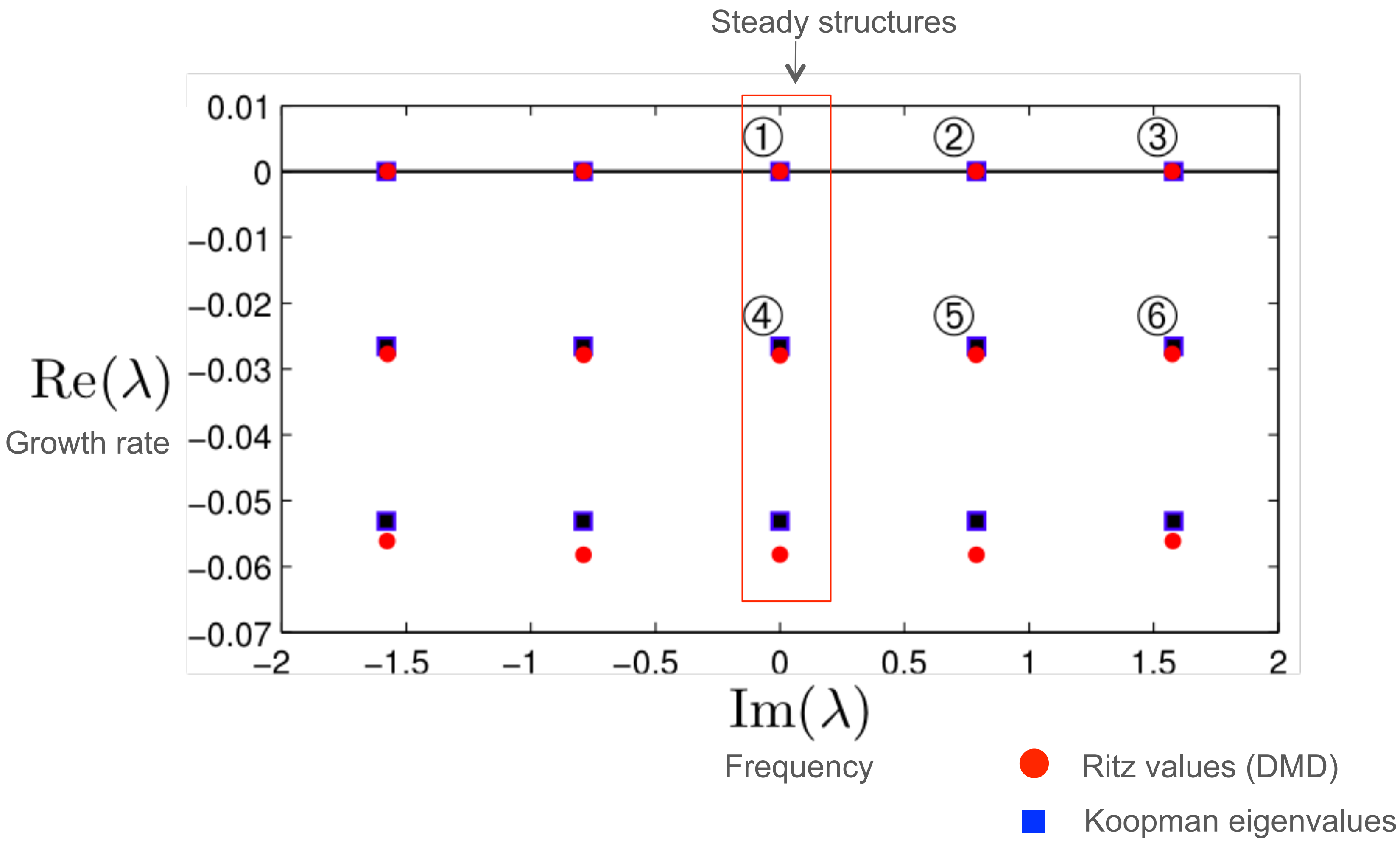
Analytical & Computational spectrum



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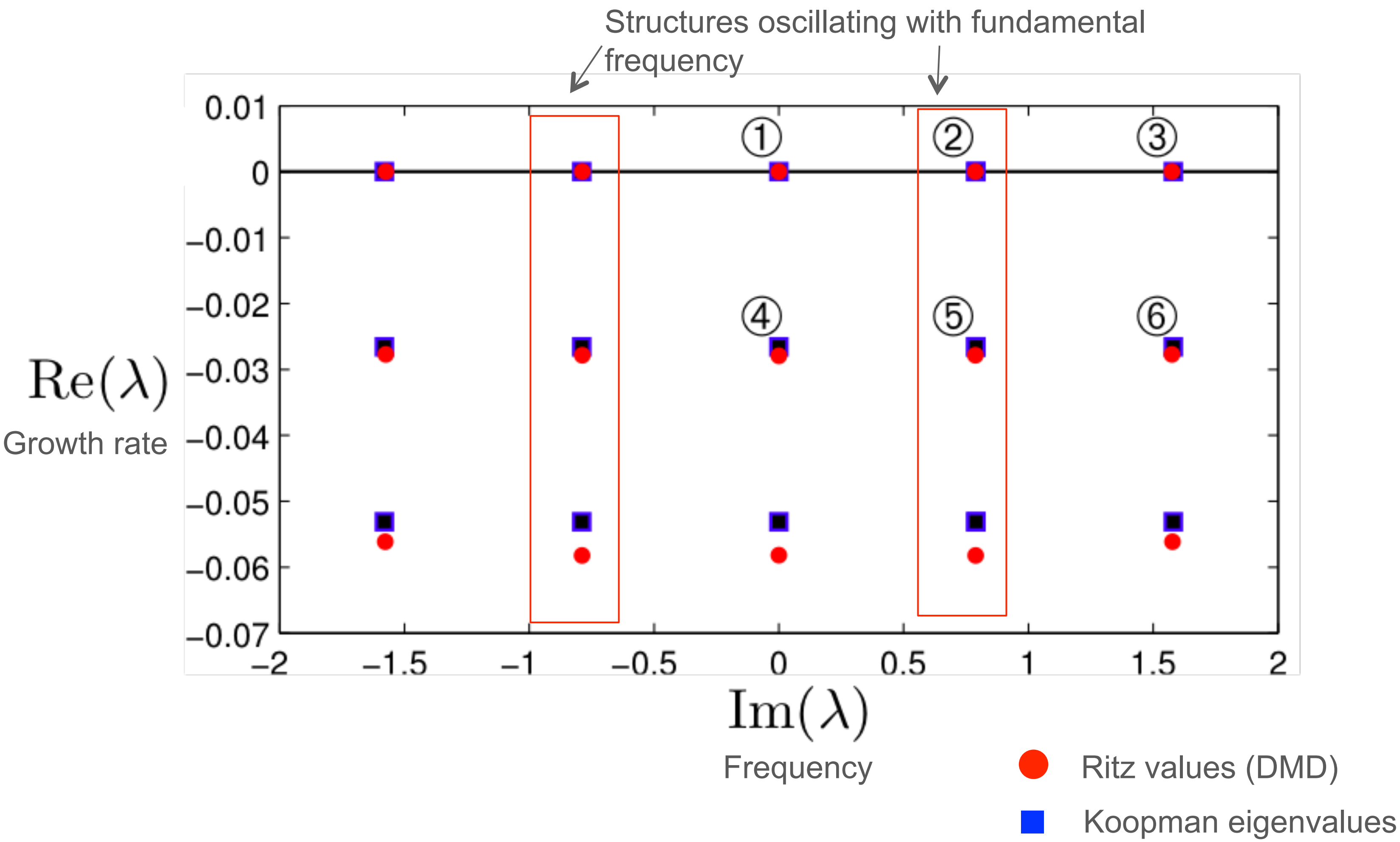
Analytical & Computational spectrum



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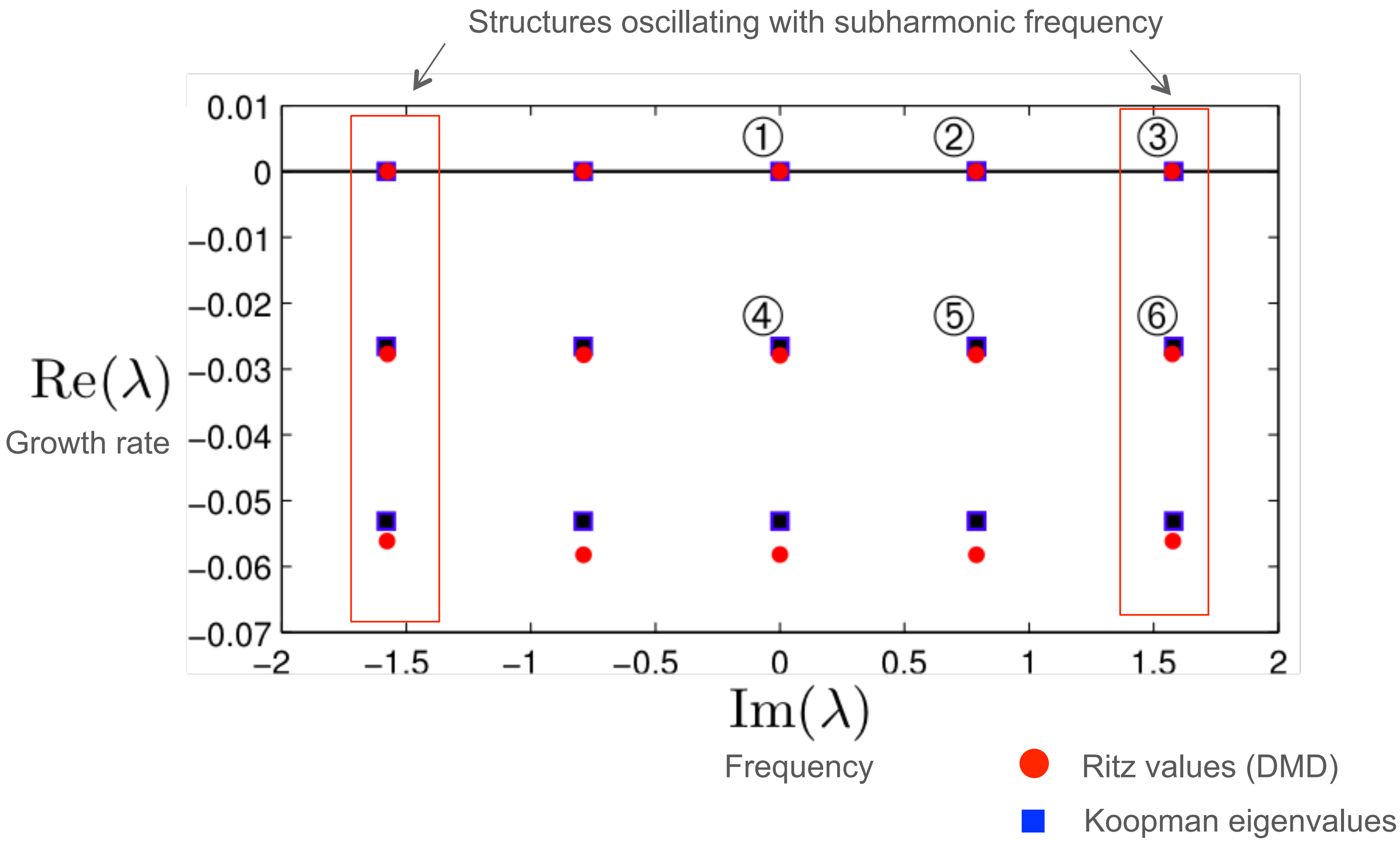
Analytical & Computational spectrum



Bagheri, 2012, Submitted JFM

Koopman modes

Expansion of an observable in Koopman eigenfunctions

$$\mathbf{g}(\mathbf{u}) = \sum_{j=0}^{\infty} \boldsymbol{\alpha}_j \phi_j(\mathbf{u}) e^{\lambda_j t} = \sum_{j=0}^{\infty} \mathbf{v}_j e^{\lambda_j t}$$

Koopman modes

$\lambda_j \in \mathbb{C}$ **Koopman eigenvalues**

Mezic, Nonlin. Dyn. 2005
Mezic, Ann. Rev. Fluid Mech. 2013

Koopman Modes

- Two steps
  1. Scale separation: fast limit cycle period but slow saturation

$$\frac{\partial A}{\partial \tau} = a_0 A - a_1 A|A|^2,$$

  2. Spectral expansion of S-L equation

$$g(A) = \sum_j \alpha_j \phi_j \exp(\tilde{\lambda}_j \tau).$$

→ Provides Koopman Modes

Bagheri, 2012, Submitted JFM

Koopman modes (analytical)

$$\mathbf{v}_{0,0} = \mathbf{u}_s + \epsilon\tilde{\mathbf{u}}_2^{(1)} + \epsilon\mu\tilde{\mathbf{u}}_2^{(2)} + \ldots,$$

$\lambda_{0,0} = 0$ — Mean flow (asymptotic)

$$\mathbf{v}_{1,0} = -\epsilon\mu\left(\frac{\mu}{r_0^2} - 1\right)\tilde{\mathbf{u}}_2^{(2)} + \ldots,$$

$\lambda_{1,0} = \sigma$ — Shift mode (transient)

$$\mathbf{v}_{0,1} = \sqrt{\mu\epsilon}\left(\frac{\sqrt{\mu}}{r_0}\right)^{i\beta} e^{i\theta_0}\left(\tilde{\mathbf{u}}_1 + \epsilon\tilde{\mathbf{u}}_3^{(1)} + \epsilon\mu\tilde{\mathbf{u}}_3^{(2)} + \ldots\right),$$

$\lambda_{0,1} = i\omega$ — Global mode (asymptotic)

$$\mathbf{v}_{1,1} = \frac{\sqrt{\mu\epsilon}}{2}\left(\frac{\sqrt{\mu}}{r_0}\right)^{i\beta}\left(\frac{\sqrt{\mu}}{r_0^2} - 1\right) e^{i\theta_0}\left((1+i\beta)(\tilde{\mathbf{u}}_1 + \epsilon\tilde{\mathbf{u}}_3^{(1)}) + \epsilon\mu(3+i\beta)\tilde{\mathbf{u}}_3^{(2)} + \ldots\right),$$

$\lambda_{1,1} = \sigma + i\omega$ — Global mode (asymptotic)

$$\mathbf{v}_{0,2} = \epsilon\mu\left(\frac{\sqrt{\mu}}{r_0}\right)^{i2\beta} e^{i2\theta_0}\tilde{\mathbf{u}}_2^{(3)} + \ldots,$$

$\lambda_{0,2} = i2\omega$ — Subharmonic mode (asymptotic)

$$\mathbf{v}_{1,2} = -\epsilon(1+i\beta)\mu\left(\frac{\sqrt{\mu}}{r_0}\right)^{i2\beta} e^{i2\theta_0}\left(\frac{\sqrt{\mu}}{r_0^2} - 1\right)\tilde{\mathbf{u}}_2^{(3)}.$$

$\lambda_{1,2} = \sigma + i2\omega$ — Subharmonic mode (transient)

Bagheri, 2012, Submitted JFM

Spectral Expansion

- Expansion of flow field into
  - Operator approach: Koopman Operator

Koopman modes

$$\mathbf{u}(t) = \sum_{j=0}^{\infty} \mathbf{v}_j e^{\lambda_j t} \qquad \lambda_j \in \mathbb{C}$$

Koopman eigenvalues

  - Computational approach - Dynamic Mode Decomposition

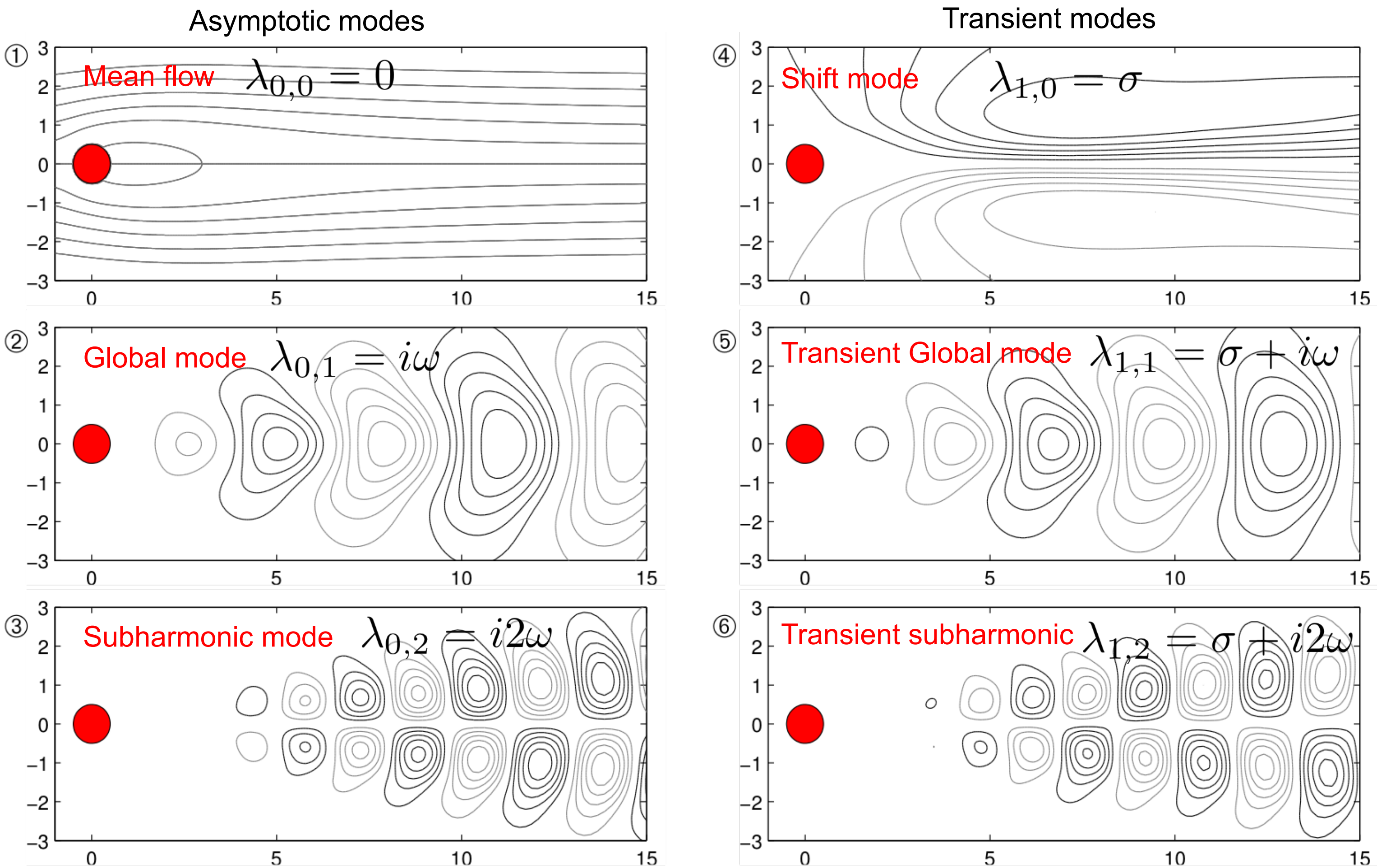
$$\mathbf{u}(t) = \sum_{j=0}^{r-1} \tilde{\mathbf{v}}_j e^{\tilde{\lambda}_j t} \qquad \tilde{\lambda}_j \in \mathbb{C}$$

Ritz values

Ritz vectors

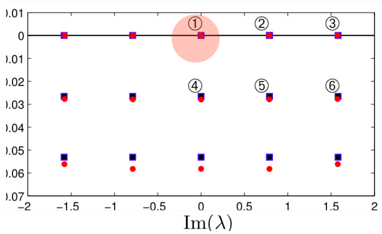
Mezic, 2005, Nonlin. Dyn.
Rowley etal 2009, JFM
Schmid, 2010, JFM

Koopman modes (computational)

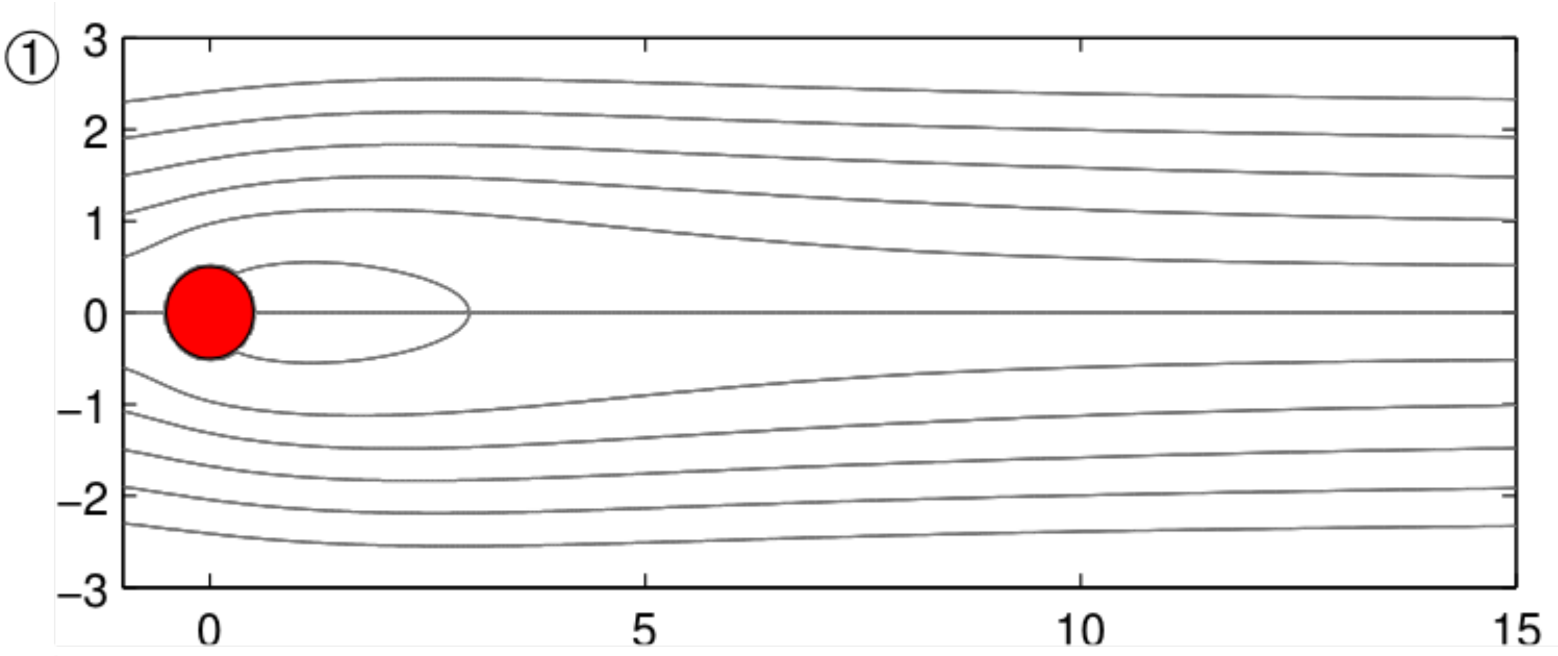


Bagheri, 2012, Submitted JFM

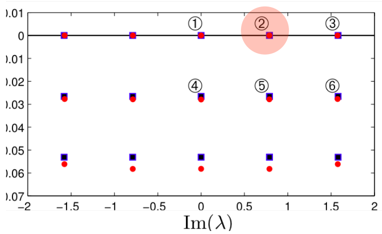
Koopman mode: Mean Flow



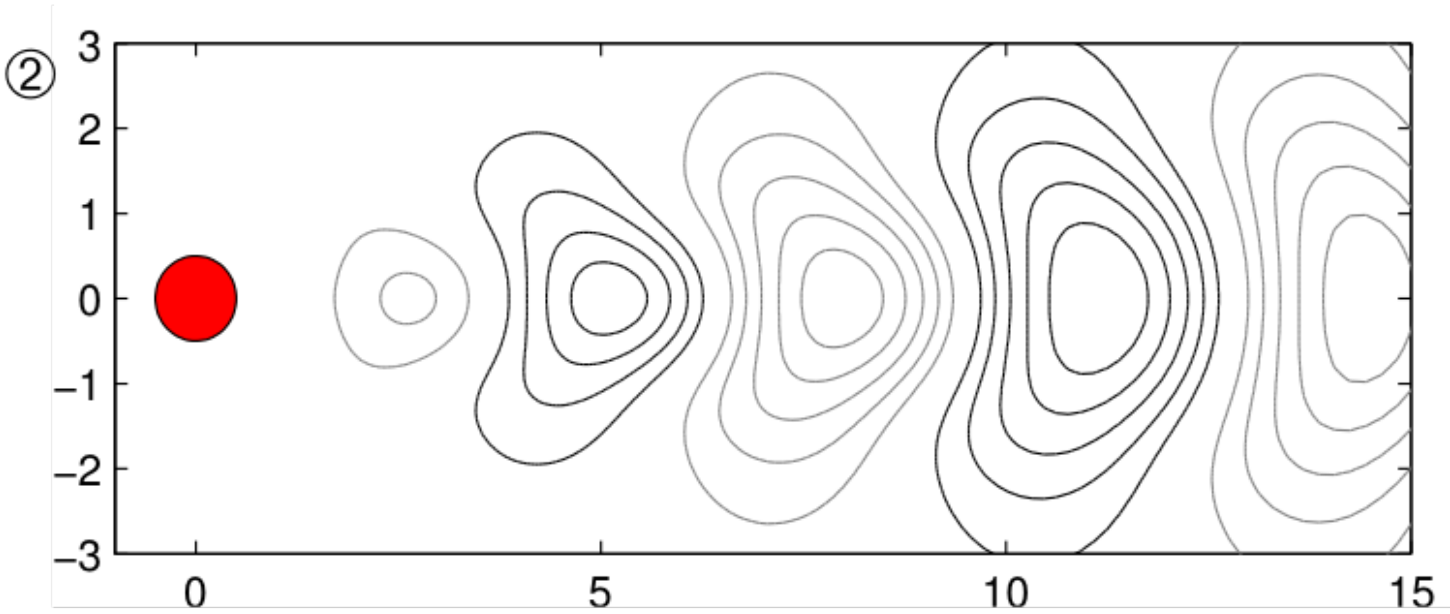
Mean flow $\lambda_{0,0} = 0$



Koopman mode: Global mode

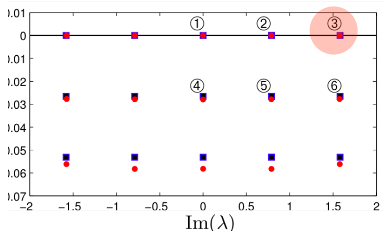


Global mode $\lambda_{0,1} = i\omega$

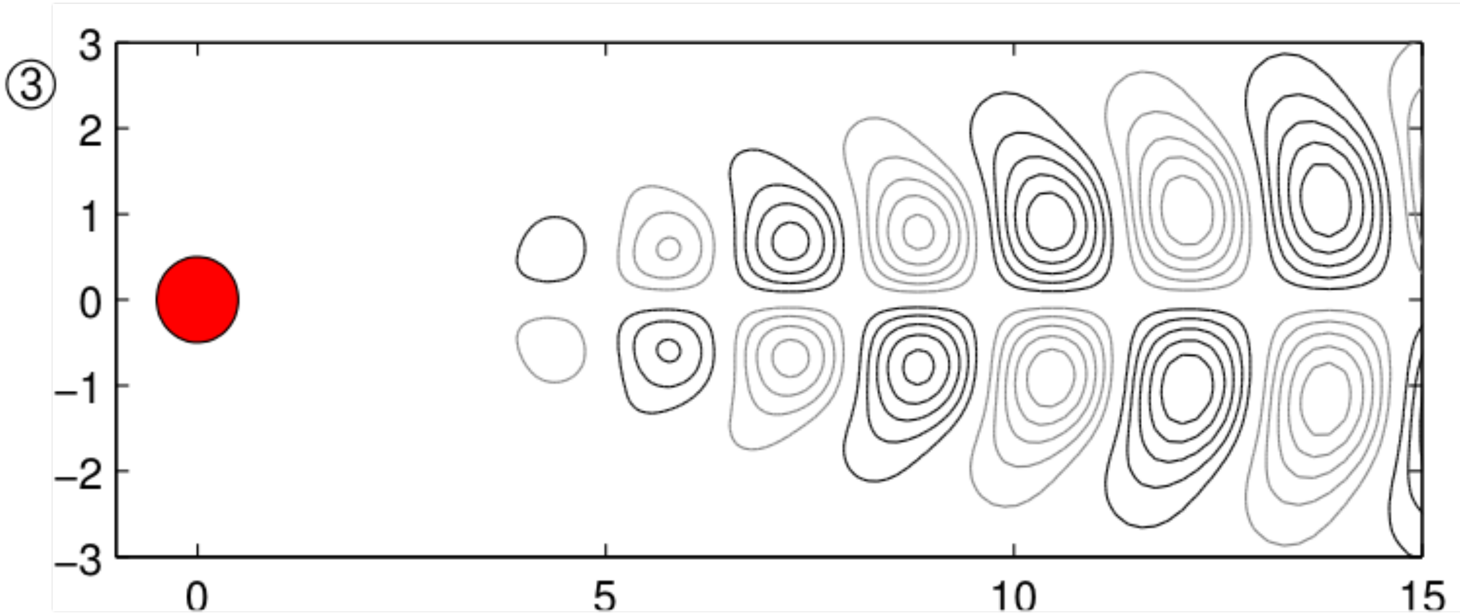


Huerre & Monkewitz, 1990, Annual Rev. Mech

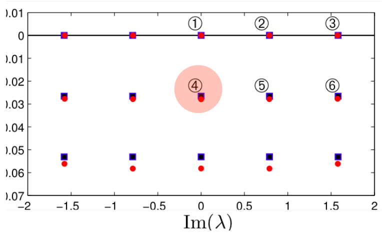
Koopman mode: Subharmonic Global mode



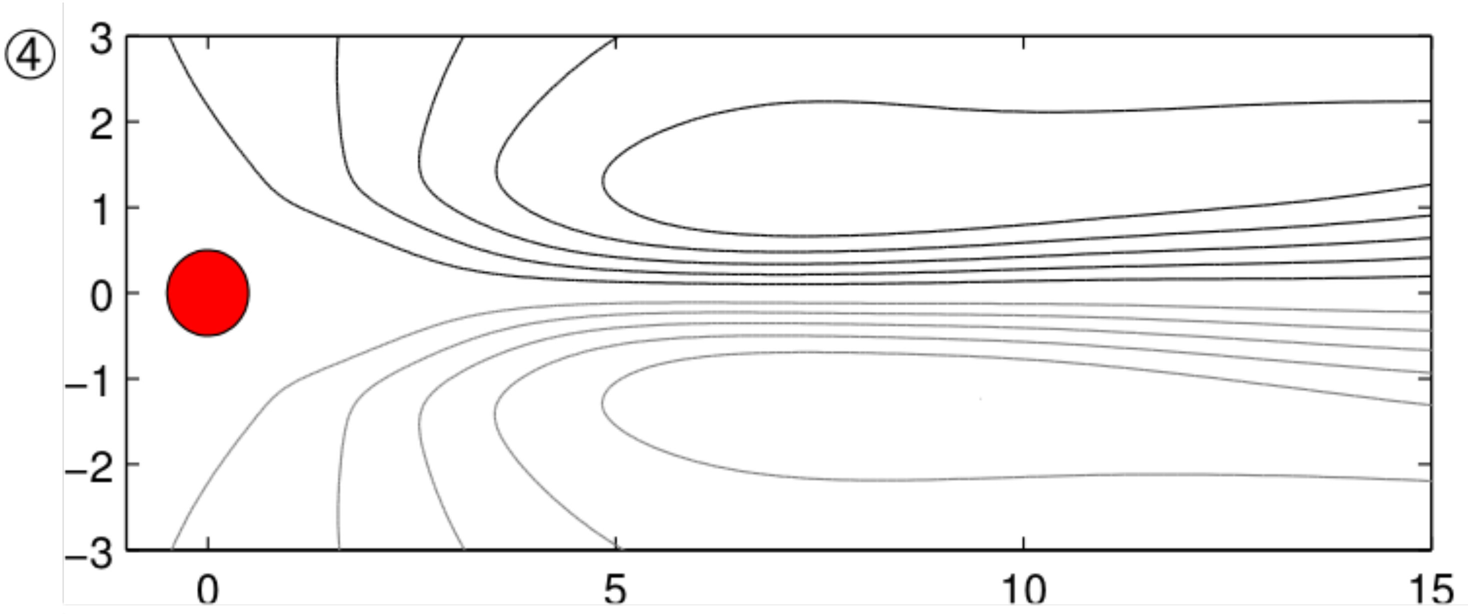
Subharmonic Global mode $\lambda_{0,2} = i2\omega$



Koopman mode: Shift mode



Shift Mode $\lambda_{1,0} = \sigma$



Noack etal, 2003, JFM

Conclusions

- Analytical results
  - Koopman eigenvalues form a lattice (trace formula)
  - Koopman modes correspond to mean flow, shift mode, global modes,..
- Computational results
  - Ritz vectors/values good approximation near and on attractor
  - Algebraic dynamics generates clusters/branches in spectrum