Effects of small noise on DMD/Koopman spectrum

Shervin Bagheri
Linné Flow Centre
Mechanics, KTH, Stockholm

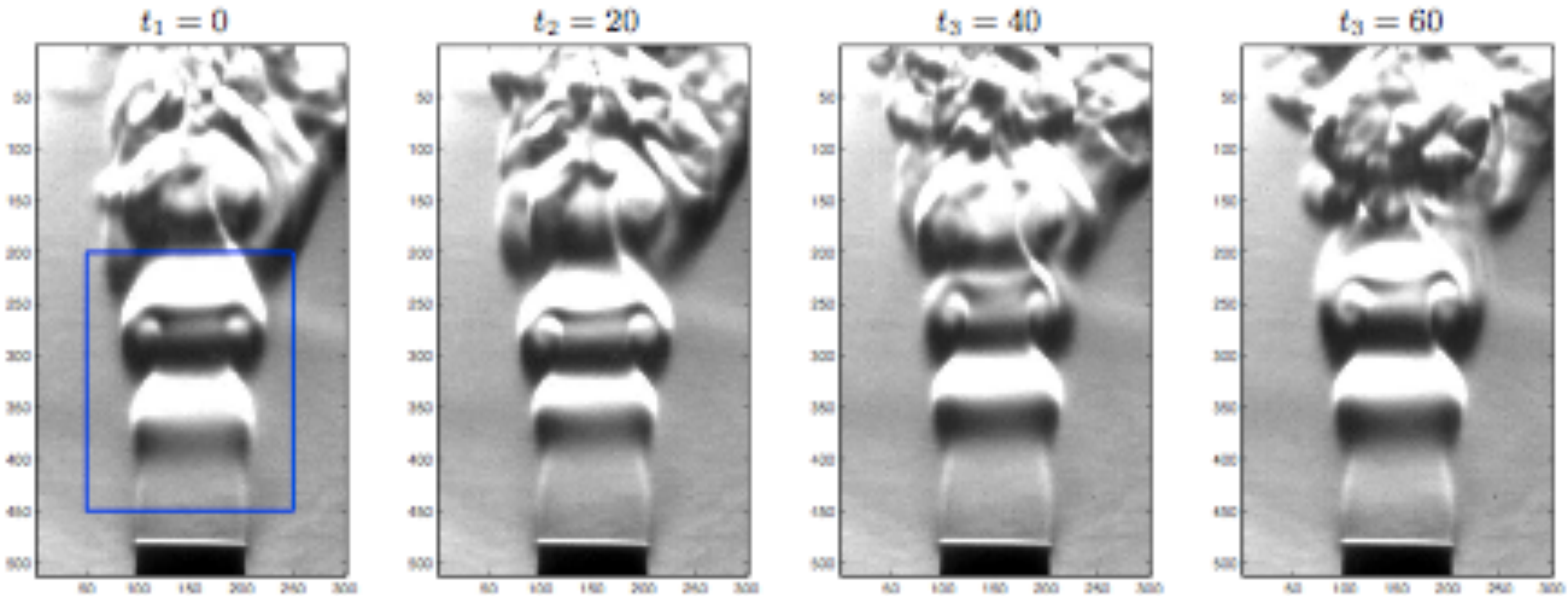
APS/DFD 2013,
Pittsburgh, Nov 24-26

Context

- A linear approach to complex non-linear system
  - Linear operators describing nonlinear flows
    (Perron-Frobenius, Koopman, Fokker-Planck, Chapman-Kolmogorov)
- Data-driven algorithms
  - Dynamic Mode Decoposition (DMD), Optimal Mode Decomposition (OMD)

DMD of Noisy System: Helium Jet

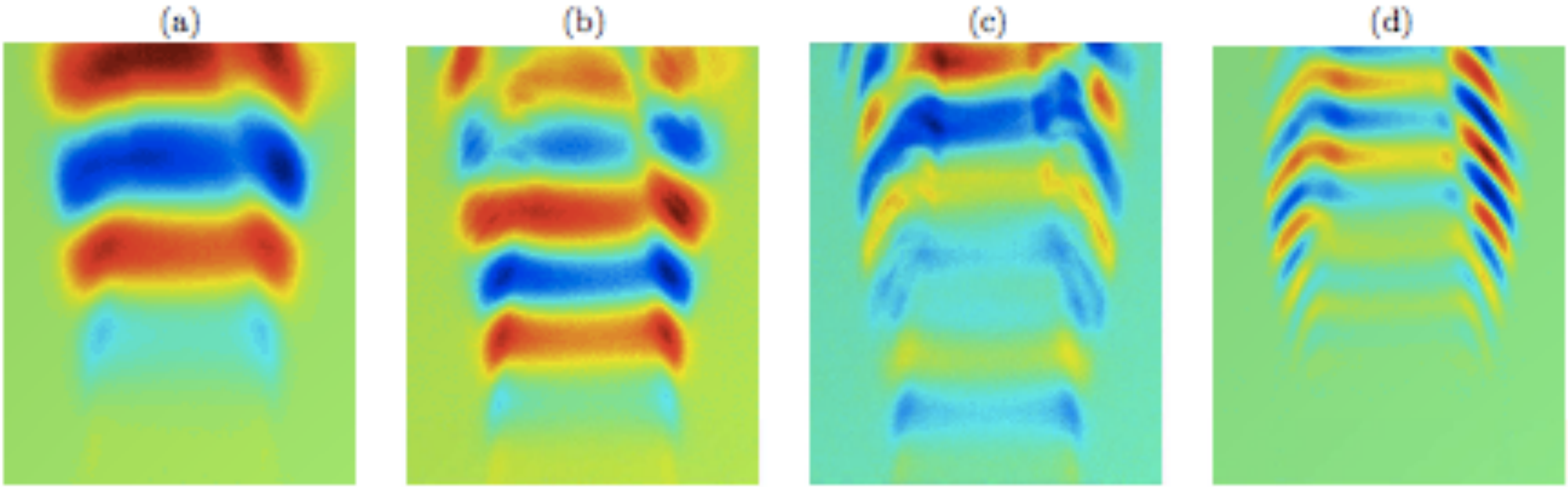
- Input: Sequence of snapshots (from experiments)



Schmid, Li, Juniper & Pust, TCFD, 2011

DMD of Noisy System: Helium Jet

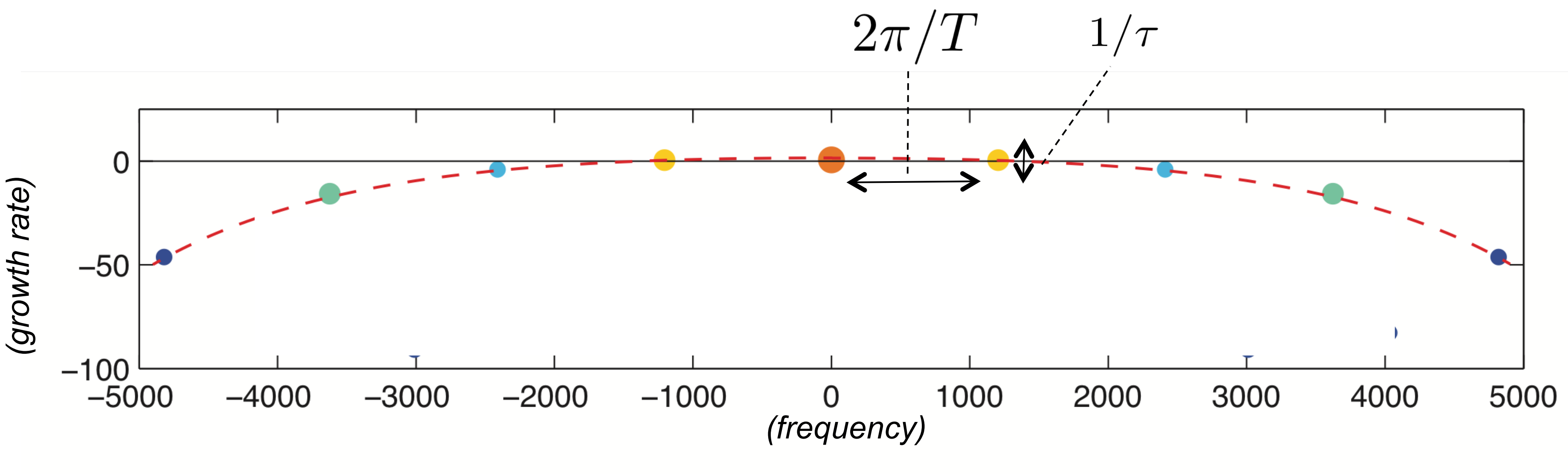
- Output: DMD modes (some "coherent structures")



Schmid, Li, Juniper & Pust, TCFD, 2011

DMD of Noisy System: Helium Jet

- Output: DMD values (growth rate and frequency)

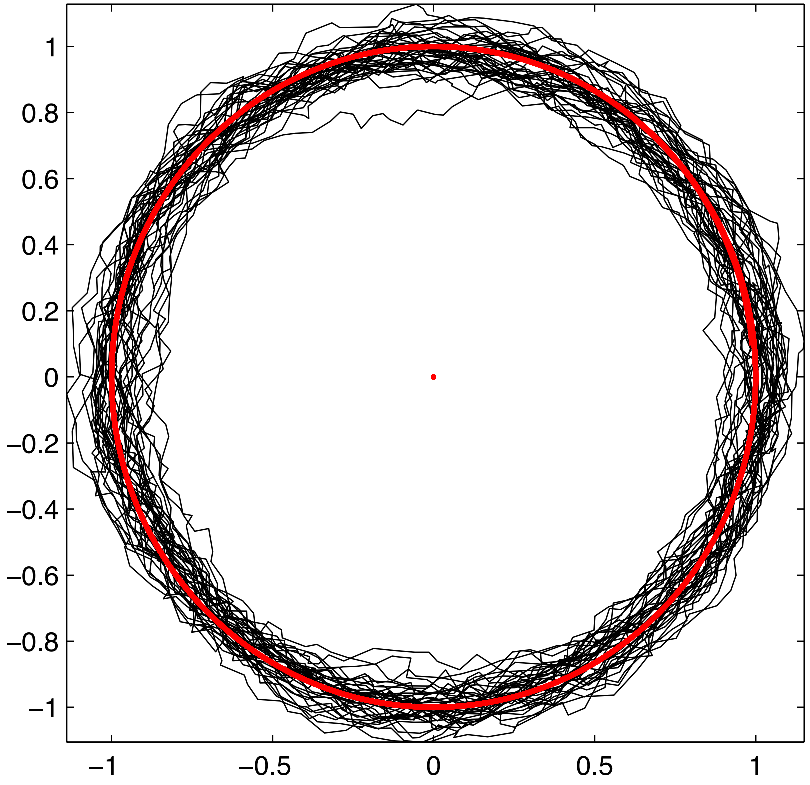


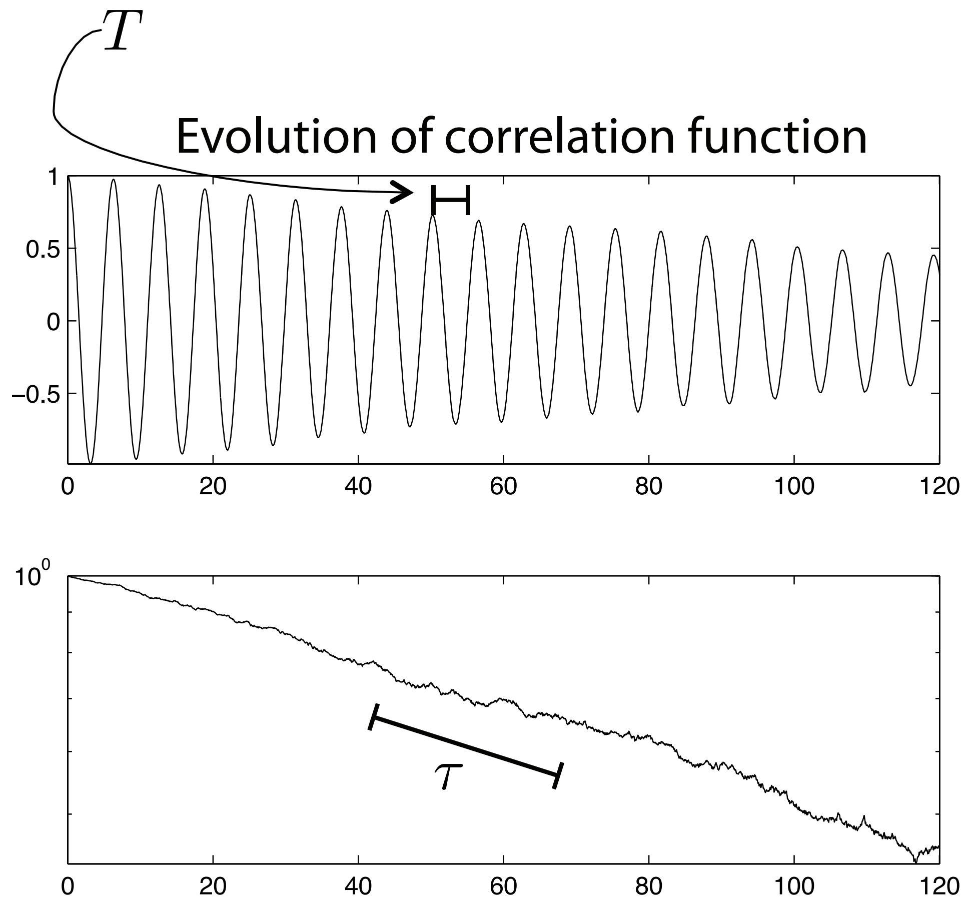
- Parabolas are due to the presence weak noise

- This DMD spectrum reveals two time scales

Effects of Noise: Limit Cycle

- Noisy limit cycle
- Deterministic limit cycle





Effects of Noise

- White noise induce a new time scale
  - Phase drift increases exponentially with rate $1/\tau$
  - Auto-correlation functions decreases exponentially with rate $1/\tau$

→ Noisy limit cycle has two time scales:

- Limit cycle period: T
- Quality factor: $Q = \dfrac{\tau}{T}$

(Q is number of oscillations which periodicity is maintained)

Stochastic Navier-Stokes Equations

- Consider noisy system

$$\dot{x} = f(x) + \underbrace{\sqrt{\epsilon}}_{\text{Noise amplitude}}\underbrace{\xi(t)}_{\text{White noise}}$$

and some observable:

$$a(x)$$

Evolution Operators

- Ensemble average

$$\langle a_t \rangle = \int a(x)\rho(x,t)dx$$

Probability density function

- Evolution governed by linear operators

$$\langle a_t \rangle = \langle a, e^{\mathcal{A}t}\rho_0 \rangle = \langle e^{\mathcal{A}^\dagger t} a, \rho_0 \rangle$$

Koopman operator

Spectral Decomposition

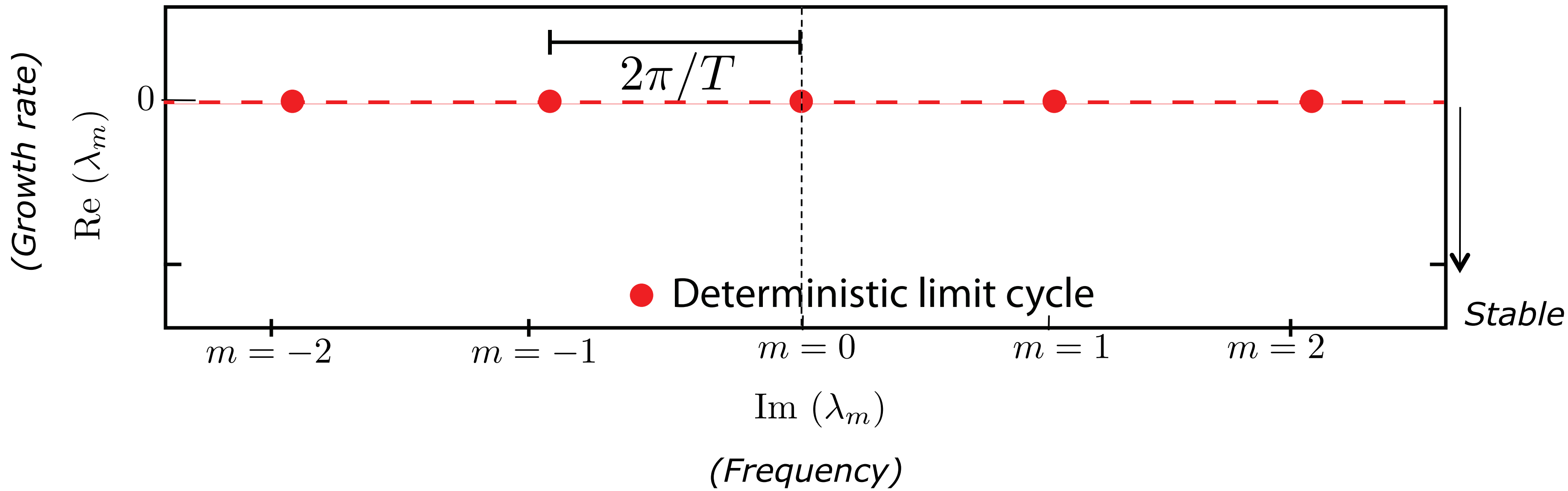
- Eigenvalue decomposition

$$\mathcal{A}\phi_m = \lambda_m \phi_m \qquad m = 0, 1, 2, \ldots$$

- Eigenvalues can be obtained from
  - WKJB expansion in noise amplitude (ϵ)
  - Transform stochastic system to deterministic Hamiltonian system of twice the size *(Onsager & Machlup, PRL, 1953)*
  - Compute trace of $\mathcal{A}$ *(Gaspard, 2002 JSP)*

Spectrum of Deterministic Limit Cycle

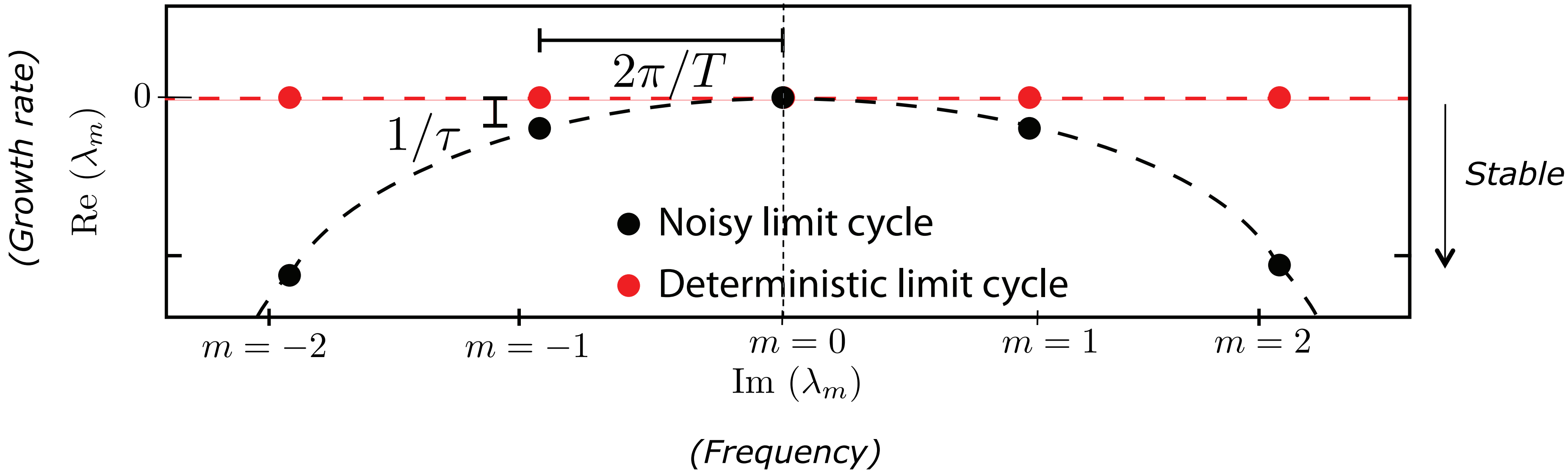
$$\lambda_m = im\frac{2\pi}{T} \qquad\qquad m = 0, 1, 2, \ldots$$



Bagheri, JFM 2013

Spectrum of Noisy Limit Cycle

$$\lambda_m = im\frac{2\pi}{T} - \underline{\epsilon m^2 \frac{|S|}{2T}\left(\frac{2\pi}{T}\right)^2} + \mathcal{O}(\epsilon^2)$$



Sensitivity

- How sensitive is a limit cycle to noise?

$$\lambda_m = im\frac{2\pi}{T} - \underline{\epsilon m^2 \frac{|S|}{2T}\left(\frac{2\pi}{T}\right)^2} + \mathcal{O}(\epsilon^2)$$

- Large $|S|$ → high rate of phase diffusion

- What is the explicit expression for S ?

Sensitivity

- One obtains the expression

$$\boxed{S = -\frac{f_1^T \delta x(T)}{f_1^T e_1}}$$

where

e_1 first Floquet vector

f_1 first adjoint Floquet vector

$\delta x(T)$ obtained by solving linearized system

Sensitivity

- One obtains the expression

$$\boxed{S = -\frac{f_1^T \delta x(T)}{f_1^T e_1}}$$

Small if system is non-normal

where

e_1 first Floquet vector

f_1 first adjoint Floquet vector

$\delta x(T)$ obtained by solving linearized system

Conclusions

- Quality factor can be read of the DMD spectrum
- Look into DMD literature: you observe parabolic branches!
- Determine whether randomness is due external noise or some intrinsic dynamics
- Noisy limit cycle may deviate from deterministic cycle if system is highly non-normal