Effects of small noise on the DMD/Koopman spectrum

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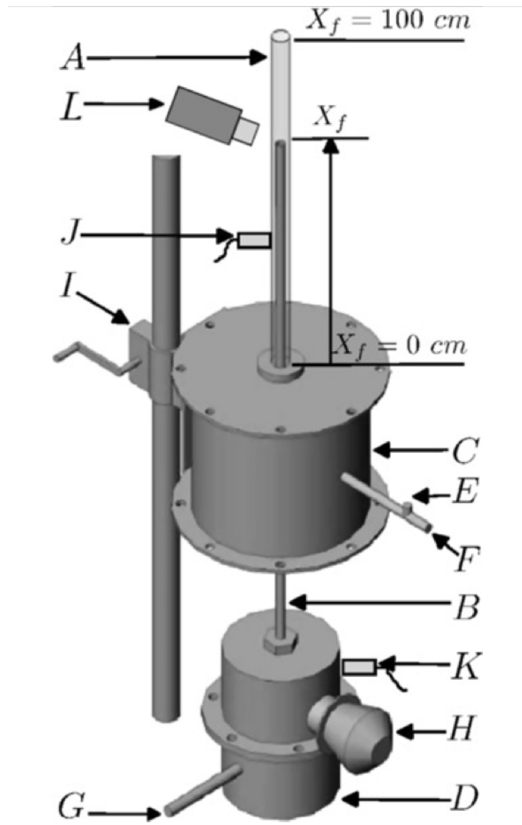
Questions

1. What are effects of noise on self-sustained oscillations?
2. How sensitive are self-sustained oscillations to noise?
3. Can we quantify the sensitivity?

Combustion Experiment

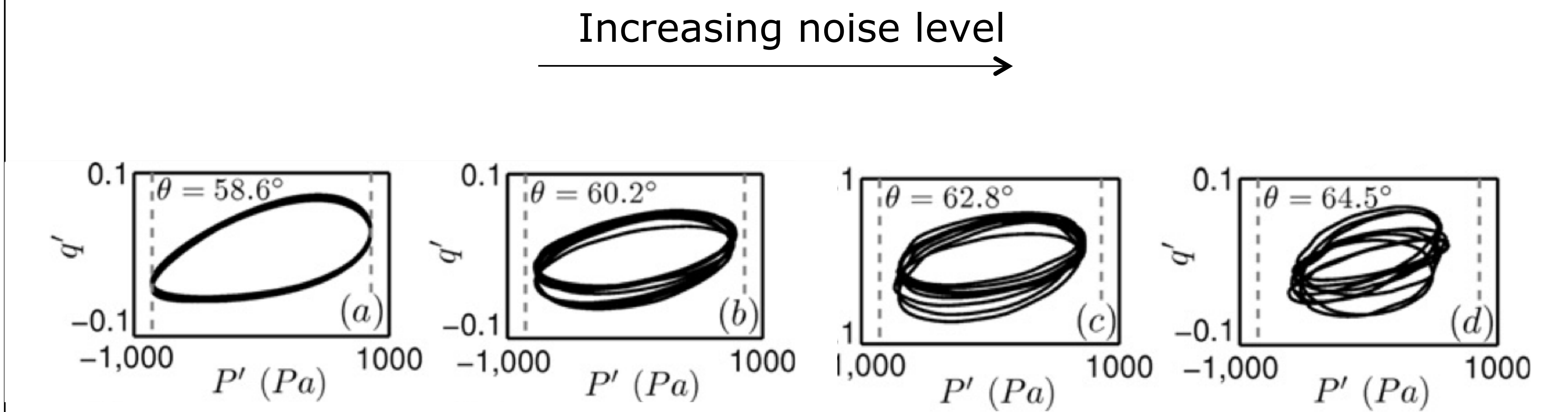
- Thermoacoustic oscillations
  - Feedback loop between acoustic field and unsteady heat release
  - White noise introduced by a loudspeaker
  - Measure pressure and heat release



Jegadeesan & Sujith, Proc. Comb. Inst. 2013

Combustion Experiment

- Noise modifies limit cycle amplitude and period



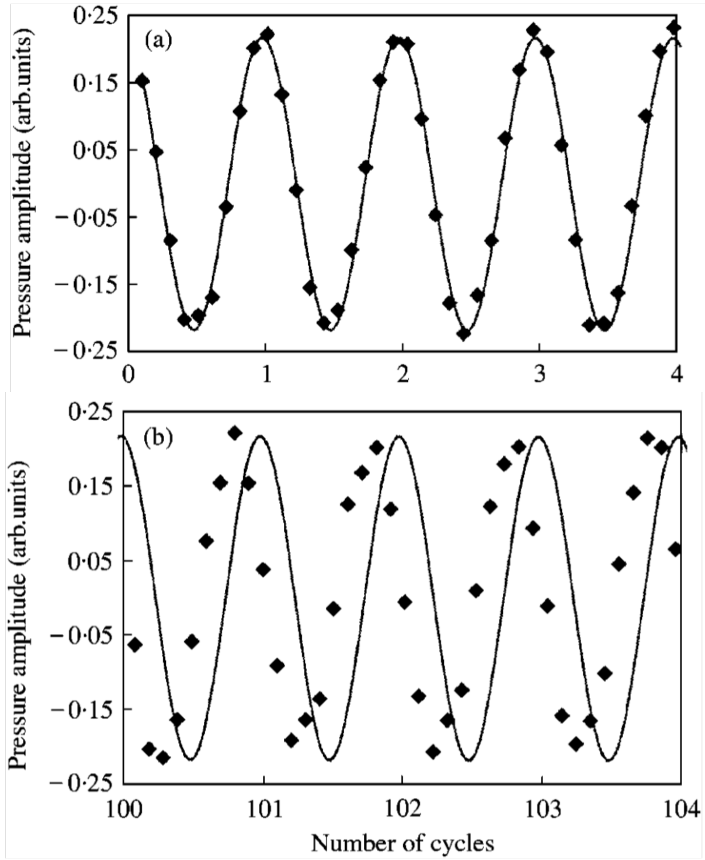
Jegadeesan & Sujith, Proc. Comb. Inst. 2013

Observations: Combustion

- Observed "drift" of the phase of pressure oscillations

  – 4 cycles:
    measuremenst in phase
    with harmonic signal

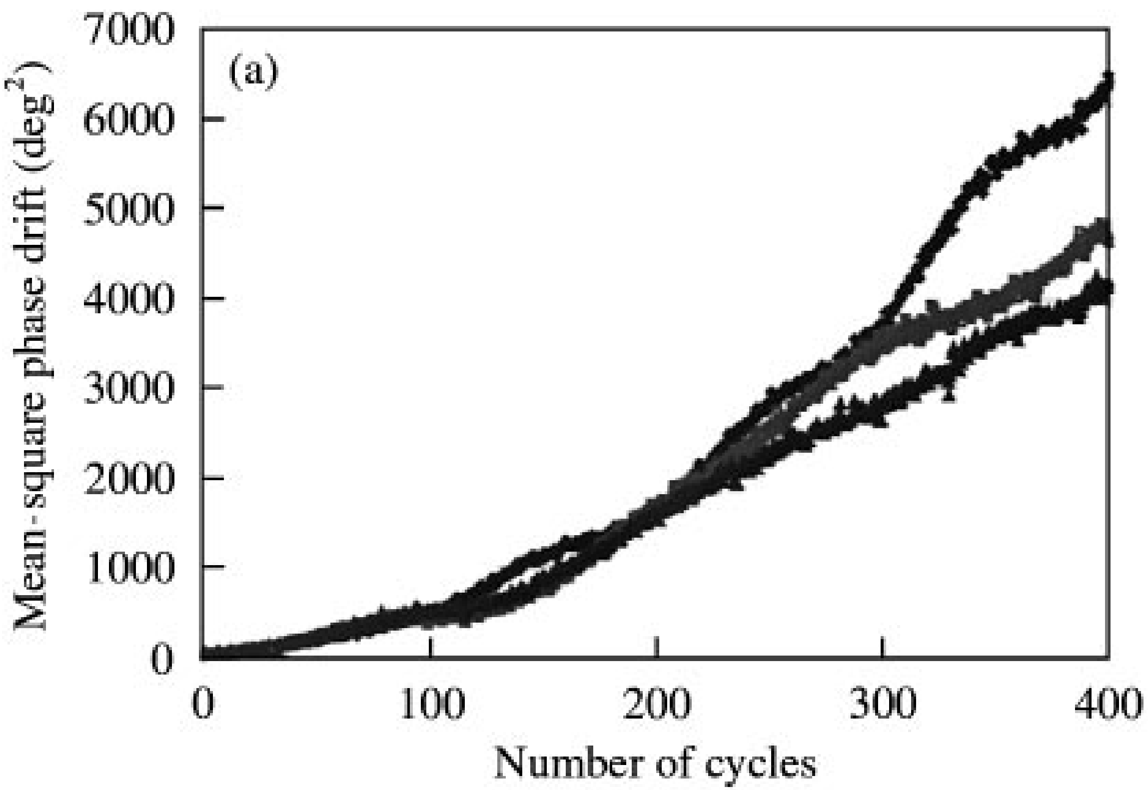
  - 100 cycles:
    phase drifted 45 degrees



Lieuwen, J. Sound & Vibration. 2001

Phase Drift

- Phase drift increases exponentially with time



Lieuwen, J. Sound & Vibration. 2001

From Experiments

- White noise induce a new time scale
  - Phase drift increases exponentially with rate $1/\tau$
  - Auto-correlation functions decreases exponentially with rate $1/\tau$
- Noisy limit cycle characterized by two time scales:
  - Limit cycle period: T
  - Quality factor: $Q = \dfrac{\tau}{T}$

(Q is number of oscillations which periodicity is maintained)

Koopman Operator

- Time evolution of observables
  - governed by Koopman operator *(Mezic, 2005, Rowley etal 2009)*
  - approximated by Dynamic Mode Decomposition (DMD) *(Schmid 2010)*
  - can be used to obtain time-correlation functions

Thus time scale τ should be present in the spectrum of the Koopman operator

Stochastic Navier-Stokes

- Consider

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{f}(\mathbf{u}) + \sqrt{\epsilon}\hat{\xi}(t)$$

  - state: $\mathbf{u} \in \mathbb{R}^n$
  - noise amplitude: ϵ
  - Gaussian white noise:

$$\langle \hat{\xi} \rangle = 0$$

$$\langle \hat{\xi}(t)\hat{\xi}(t') \rangle = Q\delta(t - t')$$

Evolution of Ensemble Average

- Given an observable $g(\mathbf{u})$

  – Ensemble average of an observable at time t

$$\langle g(\mathbf{u}) \rangle_t = \int g(\mathbf{u}) \rho(\mathbf{u}, t) d\mathbf{u}$$

$\rho(\mathbf{u}, t) d\mathbf{u}$: probability that a state will found in $d\mathbf{u}$

  – Write ensemble average with linear evolution operators:

$$\langle g(\mathbf{u}) \rangle_t = \langle g(\mathbf{u}), \mathcal{L}_t^\dagger \rho_0(\mathbf{u}) \rangle = \langle \mathcal{L}_t g(\mathbf{u}), \rho_0(\mathbf{u}) \rangle$$

Koopman operator

Eigenvalues of Koopman Operator

- Koopman operator is linear
  - Eigenvalue decomposition:

$$\mathcal{L}_t g_j^*(\mathbf{u}) = \lambda_j g_j^*(\mathbf{u}), \qquad j = 0, 1, 2, \ldots$$

  - At leading order the trace formula gives values for limit cycle:

(Gaspard, J. Stat. Phys. 2002)

$$\boxed{\lambda_m = im\omega - \epsilon \frac{|S|}{2T} \omega^2 m^2 + \mathcal{O}(\epsilon^2)}$$

$$m = 0, \pm 1, \pm 2, \ldots$$

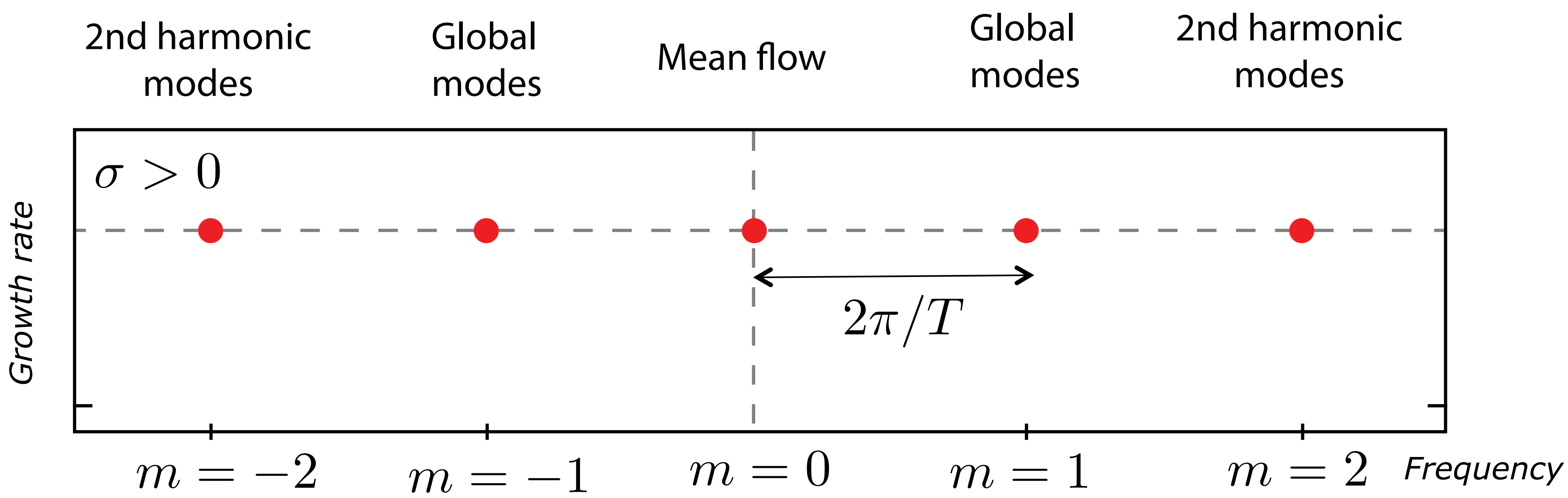
$$\omega = \frac{2\pi}{T}$$

$$S = \text{sensitivity}$$

Koopman eigenvalues for a noisy limit cycle

Koopman Spectrum

– Deterministic Koopman eigenvalues: $\lambda_m = im\omega = im\frac{2\pi}{T}$



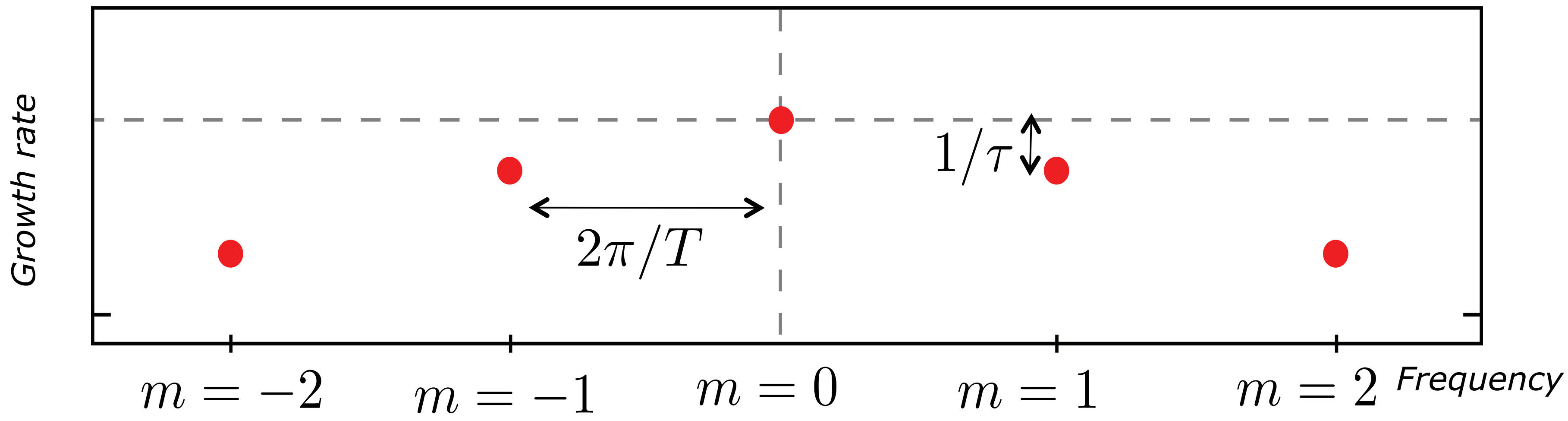
Bagheri, JFM 2013

Koopman Spectrum

- Koopman eigenvalues in the presence of noise

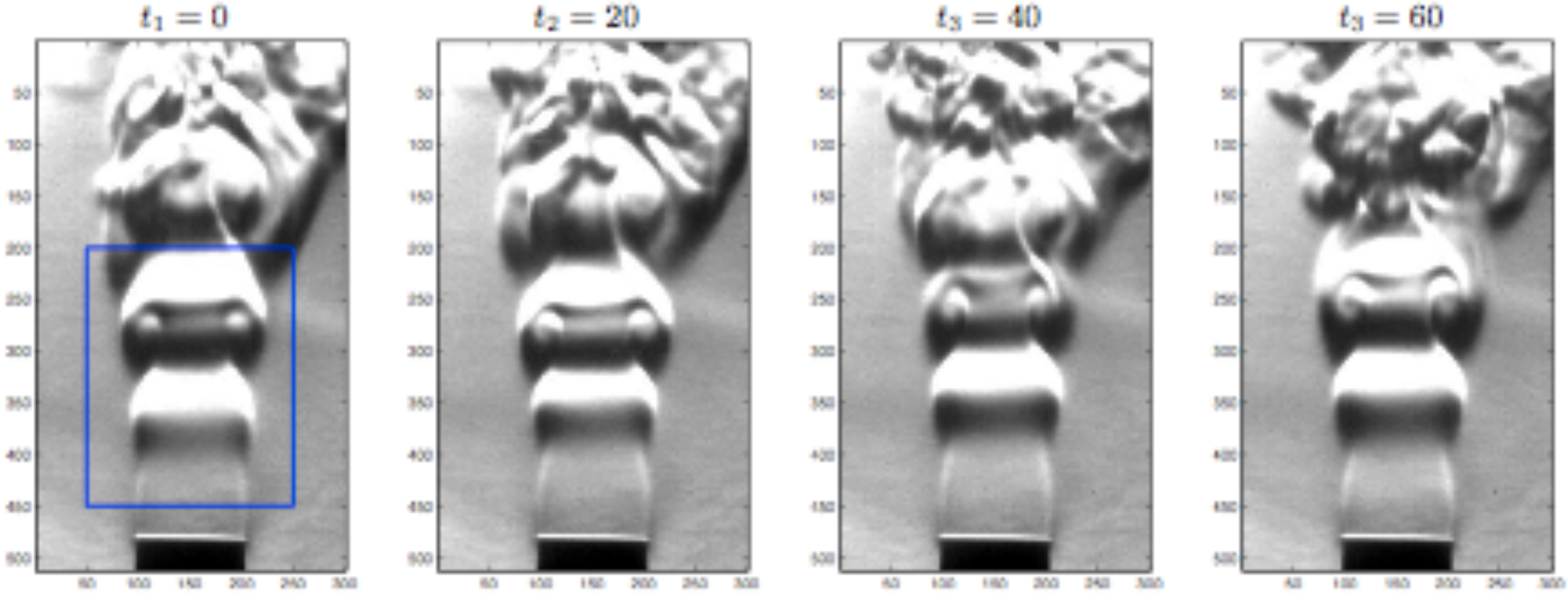
$$\lambda_m = im\omega \boxed{- \epsilon \frac{|S|}{2T}\omega^2 m^2} + \mathcal{O}(\epsilon^2) \qquad m = 0, \pm 1, \pm 2, \ldots$$

  - Non-stationary eigenvalues are damped!
  - New time scale: τ (rate of phase diffusion)
  - Proportional to S (sensitivity)



DMD of Noisy System: Helium Jet

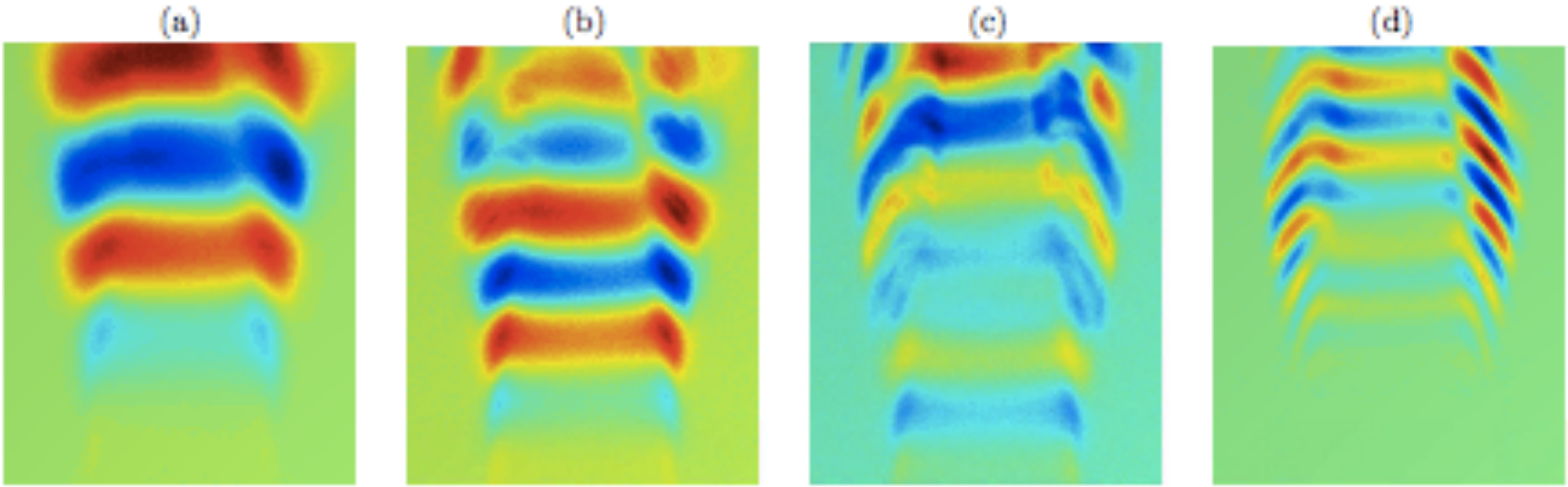
- DMD (often) approximates Koopman eigenvalues
- Input: Sequence of snapshots (from experiments)



Schmid, Li, Juniper & Pust, TCFD, 2011

DMD of Noisy System: Helium Jet

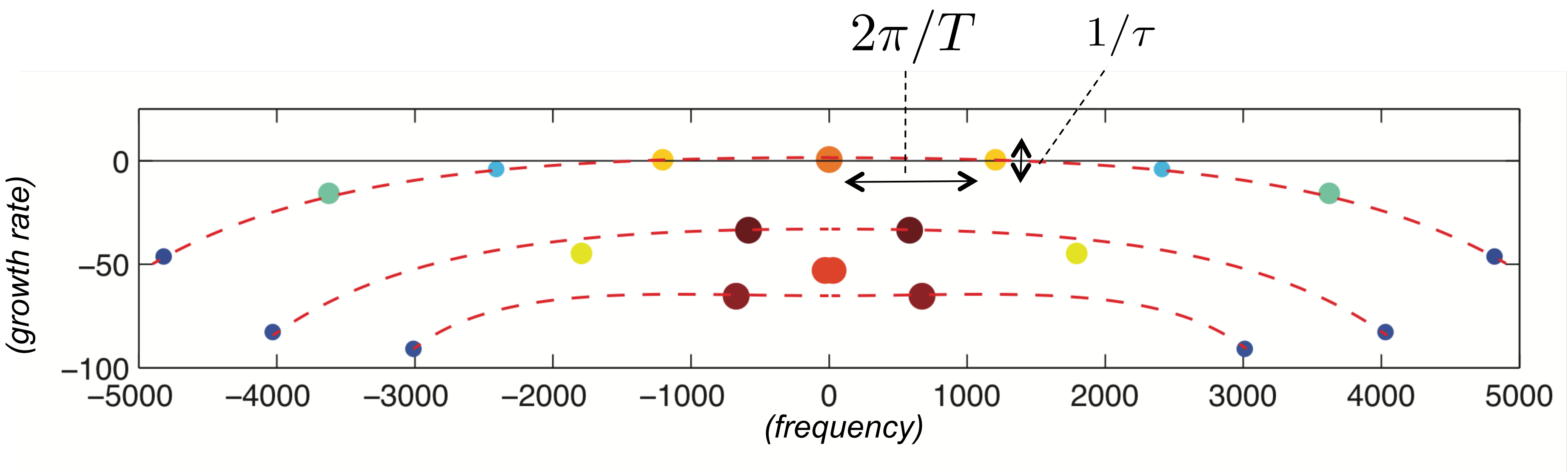
- Output: DMD modes (coherent structures)



Schmid, Li, Juniper & Pust, TCFD, 2011

DMD of Noisy System: Helium Jet

- Output: DMD values (growth rate and frequency)



- Parabolas are due to the presence weak noise

- DMD spectrum reveals two time scales

Sensitivity

- How sensitive is a limit cycle to noise?

$$\lambda_m = im\omega - \underline{\epsilon \frac{|S|}{2T} \omega^2 m^2} + \mathcal{O}(\epsilon^2)$$

- Large $|S|$ → high rate of phase diffusion

- What is the explicit expression for S ?

Stochastic Navier-Stokes

- The stochastic system

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{f}(\mathbf{u}) + \sqrt{\epsilon}\hat{\xi}(t) \qquad \mathrm{dim} = n$$

can in limit of $\epsilon \to 0$ be written as

$$\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u}) + 2Q\mathbf{p}$$
$$\dot{\mathbf{p}} = \left[-\nabla \mathbf{f}(\mathbf{u})\right]^T \mathbf{p} \qquad \mathrm{dim} = 2n$$

– where $\mathbf{p}(t) \in \mathbb{R}^n$ is an “adjoint variable”
– $\mathbf{p} = 0$ → noiseless system
– $\mathbf{p} \neq 0$ → system with noise

Onsager & Machlup, Phys. Rev. 1953

Hamiltonian System

- Hamiltonian: $H(\mathbf{u},\mathbf{p}) = \mathbf{f}(\mathbf{u})\mathbf{p} + \mathbf{p}^T Q\mathbf{p}$

  – Hamilton equations: $\dot{\mathbf{u}} = +\dfrac{\partial H}{\partial \mathbf{p}}$ $\qquad$ $\dot{\mathbf{p}} = -\dfrac{\partial H}{\partial \mathbf{u}}$

  – "Energy" conserved along trajectories (constant of motion)

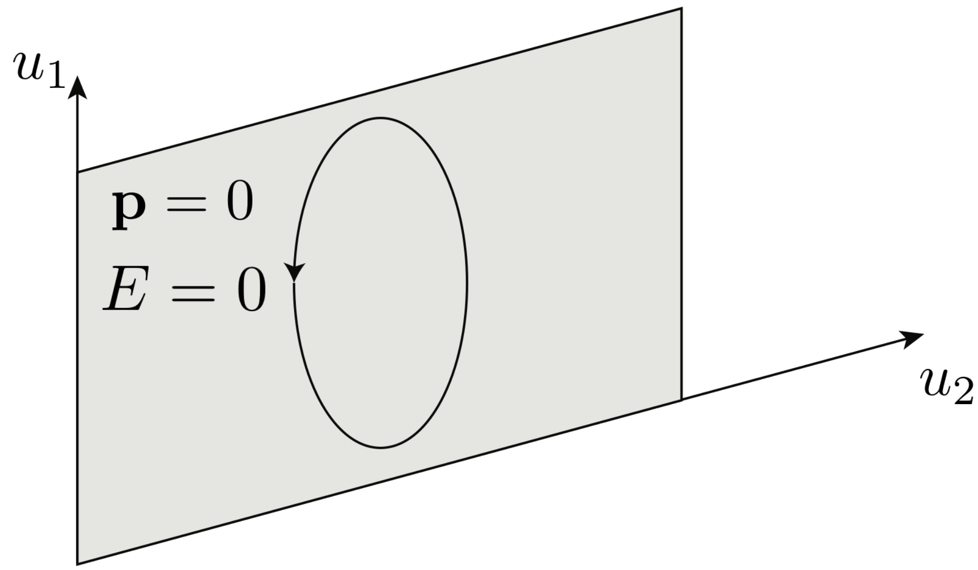
$$\frac{dH}{dt} = 0 \qquad \rightarrow \qquad H(\mathbf{u},\mathbf{p}) = E$$

  – "Energy" is measure of deviation of noisy trajectory from its deterministic prediction

Onsager & Machlup, Phys. Rev. 1953

System with One Limit Cycle

- Noise-less limit cycle



  - Period T

$$\mathbf{u}(T) = \mathbf{u}(0)$$

  - “Energy”

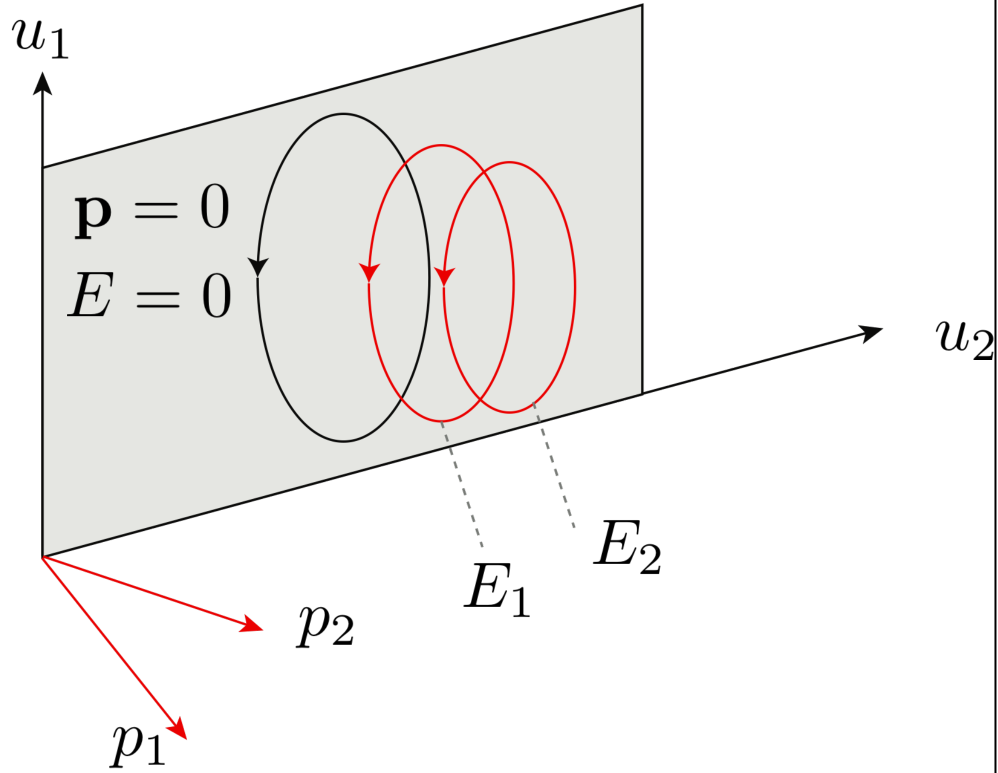
$$E = 0$$

  - The original state-space is now the subspace $\mathbf{p} = 0$

System with Noisy Limit Cycle

- Noisy limit cycle of period $T + \delta T$

$$\mathbf{u}(T + \delta T) = \mathbf{u}(0)$$
$$\mathbf{p}(T + \delta T) = \mathbf{p}(0)$$



- "Energy"

$$E = \delta E$$

  - Dimension of original state space is doubled to accommodate for limit cycles of perturbed period

Sensitivity

- Consider "Energy"

$$E = \epsilon/\tau$$

- How does a small perturbation of "energy" affect the time period?

$$E \rightarrow E + \delta E$$

$$T \rightarrow T + \delta T$$

  - define sensitivity as

$$\delta T = \left(\frac{\partial T}{\partial E} \right) \delta E = S \delta E$$

  - $\partial_E T$ large → a small perturbation of "energy", gives large deviation from deterministic time period

Linearized System

- Small δT we may linearize system

$$\delta\dot{\mathbf{u}} = \mathbf{A}(\mathbf{u}) + 2Q\delta\mathbf{p}$$

$$\delta\dot{\mathbf{p}} = -\mathbf{A}^T(\mathbf{u})\delta\mathbf{p}$$

Write solution as

$$\delta\mathbf{u}(T) = \mathbf{M}_T\delta\mathbf{u}(0) + \mathbf{N}_T\delta\mathbf{p}(0)$$

$$\delta\mathbf{p}(T) = (\mathbf{M}_T^{-1})^T\delta\mathbf{p}(0)$$

Floquet expansion

$$\mathbf{M}_T = \sum_{k=1}^{n} \phi_k \Lambda_k \psi_k^T$$

Sensitivity

- One obtains the expression

$$\boxed{\frac{\partial T}{\partial E} = -\frac{\psi_1^T \delta \mathbf{u}(T)}{\psi_1^T \phi_1}}$$

if system is non-normal then

$$\psi_1^T \phi_1 \ll 1$$

and

$$S = \frac{\partial T}{\partial E} \gg 1$$

Sensitivity

- One obtains the expression

$$\boxed{\frac{\partial T}{\partial E} = -\frac{\psi_1^T \delta \mathbf{u}(T)}{\psi_1^T \phi_1}}$$

$\delta \mathbf{u}(T)$ is obtained by solving linearized equations with initial conditions:

$$\delta \mathbf{u}(0) = 0 \qquad \delta \mathbf{p}(0) = \psi_1$$

(Gaspard, J. Stat. Phys. 2002)

Conclusions

- Sensitivity
  - Noisy limit cycle may deviate from deterministic cycle if system is highly non-normal
- Quality factor can be read of the DMD spectrum
  - Look into DMD literature: you observe parabolic branches!
  - Important for control purposes
  - Determine whether randomness is due external noise or some intrinsic dynamics