Effects of small noise on the DMD/Koopman spectrum

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Questions

1. What are effects of noise on self-sustained oscillations?
2. How sensitive are self-sustained oscillations to noise?
3. Can we quantify the sensitivity?
Combustion Experiment

- Thermoacoustic oscillations
  - Feedback loop between acoustic field and unsteady heat release
  - White noise introduced by a loudspeaker
  - Measure pressure and heat release

Combustion Experiment

- Noise modifies limit cycle amplitude and period

Increasing noise level

Observations: Combustion

- Observed “drift” of the phase of pressure oscillations
  - 4 cycles: measurement in phase with harmonic signal
  - 100 cycles: phase drifted 45 degrees

Lieuwen, J. Sound & Vibration. 2001
Phase Drift

- Phase drift increases exponentially with time

Figure 7. Dependence of the mean-squared phase drift upon the number of cycles for three different combustor operating conditions and $k$ values of (a) 30, (b) 75, and (c) 150. In all three subplots, the data were obtained at a mean inlet velocity of 14.2 m/s (top curve), 14.5 m/s (middle curve), and 14.6 m/s (bottom curve).

Lieuwen, J. Sound & Vibration. 2001
From Experiments

- White noise induce a new time scale
  - Phase drift increases exponentially with rate $1/\tau$
  - Auto-correlation functions decreases exponentially with rate $1/\tau$

- Noisy limit cycle characterized by two time scales:
  - Limit cycle period: $T$
  - Quality factor: $Q = \frac{\tau}{T}$

(Q is number of oscillations which periodicity is maintained)
Koopman Operator

- Time evolution of observables
  - governed by Koopman operator \((\text{Mezic, 2005, Rowley et al 2009})\)
  - approximated by Dynamic Mode Decomposition (DMD) \((\text{Schmid 2010})\)
  - can be used to obtain time-correlation functions

Thus time scale \(\tau\) should be present in the spectrum of the Koopman operator
Stochastic Navier-Stokes

- Consider

\[
\frac{\partial u}{\partial t} = f(u) + \sqrt{\epsilon} \hat{\xi}(t)
\]

- state: \( u \in \mathbb{R}^n \)

- noise amplitude: \( \epsilon \)

- Gaussian white noise:

  \[
  \langle \hat{\xi} \rangle = 0
  \]

  \[
  \langle \hat{\xi}(t)\hat{\xi}(t') \rangle = Q\delta(t - t')
  \]
Evolution of Ensemble Average

- Given an observable \( g(u) \)
  - Ensemble average of an observable at time \( t \)
    \[
    \langle g(u) \rangle_t = \int g(u) \rho(u, t) du
    \]
    \( \rho(u, t) du \): probability that a state will found in \( du \)
  - Write ensemble average with linear evolution operators:
    \[
    \langle g(u) \rangle_t = \langle g(u), \mathcal{L}_t^\dagger \rho_0(u) \rangle = \langle \mathcal{L}_t g(u), \rho_0(u) \rangle
    \]
    \( \mathcal{L}_t \) Koopman operator
Eigenvalues of Koopman Operator

- Koopman operator is linear
  - Eigenvalue decomposition:
    \[ \mathcal{L}_t g^*_j(u) = \lambda_j g^*_j(u), \quad j = 0, 1, 2, \ldots \]
  - At leading order the trace formula gives values for limit cycle:
    \[ \lambda_m = im\omega - \epsilon \frac{|S|}{2T} \omega^2 m^2 + O(\epsilon^2) \]

\[ m = 0, \pm 1, \pm 2, \ldots \]
\[ \omega = \frac{2\pi}{T} \]
\[ S = \text{sensitivity} \]

Koopman eigenvalues for a noisy limit cycle

Koopman Spectrum

- Deterministic Koopman eigenvalues: $\lambda_m = im\omega = im\frac{2\pi}{T}$

2nd harmonic modes | Global modes | Mean flow | Global modes | 2nd harmonic modes

$\sigma > 0$

$m = -2$ $m = -1$ $m = 0$ $m = 1$ $m = 2$

$2\pi/T$
Koopman Spectrum

- Koopman eigenvalues in the presence of noise

\[ \lambda_m = im\omega - \epsilon \frac{|S|}{2T} \omega^2 m^2 + \mathcal{O}(\epsilon^2) \quad m = 0, \pm 1, \pm 2, \ldots \]

- Non-stationary eigenvalues are damped!
- New time scale: \( \tau \) (rate of phase diffusion)
- Proportional to \( S \) (sensitivity)
DMD of Noisy System: Helium Jet

- DMD (often) approximates Koopman eigenvalues
- Input: Sequence of snapshots (from experiments)

Schmid, Li, Juniper & Pust, TCFD, 2011
DMD of Noisy System: Helium Jet

- Output: DMD modes (coherent structures)

Schmid, Li, Juniper & Pust, TCFD, 2011
DMD of Noisy System: Helium Jet

- Output: DMD values (growth rate and frequency)

- Parabolas are due to the presence of weak noise

- DMD spectrum reveals two time scales

\[
\frac{2\pi}{T}, \quad \frac{1}{\tau}
\]
Sensitivity

- How sensitive is a limit cycle to noise?

\[ \lambda_m = im\omega - \epsilon \frac{|S|}{2T} \omega^2 m^2 + \mathcal{O}(\epsilon^2) \]

- Large \( |S| \) \( \rightarrow \) high rate of phase diffusion

- What is the explicit expression for \( S \) ?
The stochastic system

\[
\frac{\partial \mathbf{u}}{\partial t} = \mathbf{f}(\mathbf{u}) + \sqrt{\epsilon} \hat{\xi}(t) \quad \text{dim} = n
\]

can in limit of \( \epsilon \to 0 \) be written as

\[
\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u}) + 2Q\mathbf{p} \quad \text{dim} = 2n
\]
\[
\dot{\mathbf{p}} = [-\nabla \mathbf{f}(\mathbf{u})]^T \mathbf{p}
\]

- where \( \mathbf{p}(t) \in \mathbb{R}^n \) is an “adjoint variable”
- \( \mathbf{p} = 0 \) → noiseless system
- \( \mathbf{p} \neq 0 \) → system with noise

Onsager & Machlup, Phys. Rev. 1953
Hamiltonian System

- Hamiltonian: \[ H(u, p) = f(u)p + p^T Qp \]

\[ \begin{align*}
\dot{u} &= + \frac{\partial H}{\partial p} \\
\dot{p} &= - \frac{\partial H}{\partial u}
\end{align*} \]

- “Energy” conserved along trajectories (constant of motion)

\[ \frac{dH}{dt} = 0 \quad \rightarrow \quad H(u, p) = E \]

- “Energy” is measure of deviation of noisy trajectory from its deterministic prediction

Onsager & Machlup, Phys. Rev. 1953
System with One Limit Cycle

- Noise-less limit cycle
  - Period \( T \)
    \[ u(T) = u(0) \]
  - “Energy”
    \[ E = 0 \]
  - The original state-space is now the subspace \( p = 0 \)
System with Noisy Limit Cycle

- Noisy limit cycle of period $T + \delta T$
  
  $\mathbf{u}(T + \delta T) = \mathbf{u}(0)$
  
  $\mathbf{p}(T + \delta T) = \mathbf{p}(0)$

- “Energy”
  
  $E = \delta E$

  - Dimension of original state space is doubled to accommodate for limit cycles of perturbed period
Sensitivity

• Consider “Energy”

\[ E = \frac{\epsilon}{\tau} \]

• How does a small perturbation of “energy” affect the time period?

\[ E \rightarrow E + \delta E \]

\[ T \rightarrow T + \delta T \]

– define sensitivity as

\[ \delta T = \left( \frac{\partial T}{\partial E} \right) \delta E = S \delta E \]

– \( \partial_E T \) large \( \rightarrow \) a small perturbation of “energy”, gives large deviation from deterministic time period
Linearized System

- Small $\delta T$ we may linearize system

\[
\begin{align*}
\delta \dot{u} &= A(u) + 2Q\delta p \\
\delta \dot{p} &= -A^T(u)\delta p
\end{align*}
\]

Write solution as

\[
\begin{align*}
\delta u(T) &= M_T\delta u(0) + N_T\delta p(0) \\
\delta p(T) &= (M_T^{-1})^T\delta p(0)
\end{align*}
\]

Floquet expansion

\[
M_T = \sum_{k=1}^{n} \phi_k \Lambda_k \psi_k^T
\]
Sensitivity

- One obtains the expression

\[
\frac{\partial T}{\partial E} = - \frac{\psi_1^T \delta u(T)}{\psi_1^T \phi_1}
\]

if system is non-normal then

\[\psi_1^T \phi_1 \ll 1\]

and

\[S = \frac{\partial T}{\partial E} \gg 1\]
Sensitivity

- One obtains the expression

\[
\frac{\partial T}{\partial E} = -\frac{\psi_1^T \delta u(T)}{\psi_1^T \phi_1}
\]

\(\delta u(T)\) is obtained by solving linearized equations with initial conditions:

\[
\delta u(0) = 0 \quad \delta p(0) = \psi_1
\]

Conclusions

- **Sensitivity**
  - Noisy limit cycle may deviate from deterministic cycle if system is highly non-normal

- **Quality factor can be read of the DMD spectrum**
  - Look into DMD literature: you observe parabolic branches!
    - Important for control purposes
    - Determine whether randomness is due external noise or some intrinsic dynamics